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by Stephen R. Blough

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The question of whether aggregate output is best described as a trend-stationary (TS) or as a difference-stationary (DS, or unit root) process continues to generate a substantial volume of research a dozen years after it was first raised by Nelson and Plosser (1982), including a recent paper by Rudebusch (1993). Rudebusch argues that "Based on the usual unit root tests, little can be said about the relative likelihood of the specific DS and TS models of real GNP." Rudebusch concludes by emphasizing "the importance of measuring the confidence intervals for estimates of persistence without conditioning on the TS or DS model."

This paper provides a strong theoretical result on distinguishing TS and DS models, and gives confidence intervals for the GNP impulse response function that do not require such distinction. Theoretically, the paper shows that, in the absence of *a priori* specification restrictions, the classes of unit root and stationary processes are *nearly observationally equivalent*: no finite data sample can provide information on the TS/DS issue. The paper then shows how the principle of parsimony for time series model specification masks near observational equivalence and implicitly rules out plausible shapes for the univariate impulse response function. Finally, the paper provides confidence intervals for the GNP impulse response function using two methods that nest parsimonious TS and DS models.

1. Near Observational Equivalence.

The classes of unit root and stationary processes are nearly observationally equivalent in that any finite sample, finite horizon, or discounted infinite horizon behavior of *any* member of *either* class of processes can be arbitrarily well approximated by members of the other class. To see this, consider processes of the form:

$$\begin{gathered}(1-\rho L)y_t = (1+\theta L)b(L)\varepsilon_t \\ \varepsilon_t = 0 \quad \forall t \le s \\ y_s \sim S(y_s) \\ -\infty < s \le 0\end{gathered} \tag{1}$$

where $\{\varepsilon_t\}$ is a white noise process, L is the lag operator, ρ and θ are each less than or equal to one in absolute value, and $b(L)$ is a lag polynomial of possibly infinite length whose roots lie entirely outside the unit circle and whose coefficients are square summable. Observations of $\{y_t\}$ are available for t=1,...,T. So that the observations have a well-defined unconditional distribution in the unit root case ρ=1, the process is assumed to have start date s with initial value y_s drawn from some proper distribution function $S(.)$. All results are unchanged if $\{y_t\}$ contains arbitrary deterministic components, such as time trends; these are omitted from the representation for notational compactness.

Representation (1) nests the Wold representations of all stationary processes (when ρ=θ=0) and of all difference stationary processes (when ρ=1 and θ=0). The process $\{y_t\}$ is stationary for all $|\rho|<1$. Fixing ρ=1, $\{y_t\}$ has a unit root for all $\theta\neq$-1, while for θ=-1 the roots cancel and $\{y_t\}$ is again stationary.

Near observational equivalence is shown by examining sequences of *processes* while keeping sample size fixed (in contrast to increasing sample size while keeping the process fixed, as in asymptotic theory). Equation (1) gives each observation y_t as a function of the parameters $(\rho,\theta,b(L))$ and the realization of the random variables $\lambda\equiv$ (y_s ; ε_t t=s+1,...,T). Denote by $\{y_t(\rho,\theta,b(L))\}_T$ a sample of length T from a process (1), with dependence on λ understood.

Then for fixed values of θ and $b(L)$, a sequence of processes is implied by a sequence of values of ρ. Consider in particular sequences of the form:

$$\begin{gathered}\left[\{y_t(\rho_i,0,b(L))\}_T\right] \quad i=1,\ldots,\infty \\ |\rho_i|<1 \quad \forall i \\ \lim_{i\to\infty}\rho_i=1\end{gathered} \tag{2}$$

Each process in this sequence is stationary, while the limiting process has a unit root. It is essential to note that the limiting process can be *any* unit root process by appropriate choice of $b(L)$. .

Conversely, fixing ρ and $b(L)$, a sequence of values of θ implies a sequence of processes. All processes in the sequence:

$$\left[\{y_t(1,\theta_i,b(L))\}_T\right] \quad i=1,\ldots,\infty$$
$$\theta_i > -1 \quad \forall i \tag{3}$$
$$\lim_{i\to\infty}\theta_i = -1$$

have a unit root. However, the limiting process is $\{y_t(1,-1,b(L))\}$, which by cancellation of the roots is the stationary process $\{y_t(0,0,b(L))\}$. Again, note that appropriate choice of $b(L)$ permits the limiting process of a sequence (3) to be *any* stationary process.

Near observational equivalence is formalized by the following proposition, which is proved in the Appendix. The proposition states that, for any statistical purpose, sequences of processes (2) and (3) converge to the limiting processes. Since the sequences hold sample size fixed, the proposition applies to exact finite sample distributions.

Proposition 1.[1] Consider any function of the data $g_T(.)$ that is defined for the finite sample size T and is continuous almost everywhere with respect to processes of the form (1).

(i) For sequences satisfying (2), for any $b(L)$, $g_T(\{y_t(\rho_i,0,b(L))\}_T)$ converges in distribution to $g_T(\{y_t(1,0,b(L))\}_T)$ as $i\to\infty$.

(ii) For sequences satisfying (3), for any $b(L)$, $g_T(\{y_t(1,\theta_i,b(L))\}_T)$ converges in distribution to $g_T(\{y_t(0,0,b(L))\}_T)$ as $i\to\infty$.

The Proposition shows that there are stationary processes that can arbitrarily well approximate any statistical implication of any unit root process, and that there are unit root processes that can arbitrarily well approximate any statistical implication of any stationary process. There are unit root processes that arbitrarily well approximate white noise (white noise plus a random walk with very small innovation variance, for example). Because *all* processes of *both* classes can be arbitrarily well approximated by processes in the other class, the two

[1]Near observational equivalence appears to have been first noted by Clark (1988). Part (ii) of this Proposition is closely related to results in Cochrane (1991) and Blough (1992a).

classes are nearly observationally equivalent.[2] No statistical procedure can meaningfully discriminate between them.[3]

Tests for a unit root in GNP (or other data series) are generally taken to provide information on "persistence" of the process. Define $\psi_j(\rho,\theta,b(L))$ to be the impulse response function of the process $\{y_t(\rho,\theta,b(L))\}$ at horizon j:

$$\psi_j(\rho,\theta,b(L)) \equiv \frac{\partial y_{t+j}(\rho,\theta,b(L))}{\partial \varepsilon_t} \tag{4}$$

TS and DS processes are distinct in that the *infinite* horizon impulse response ψ_∞ is zero for TS processes and non-zero for DS processes. The effect of a shock over long *finite* horizons may well have economic interest. Present value models often make use of the present discounted value of the impulse response function to an infinite horizon $\Psi(\rho,\theta,b(L);\beta)$, defined for a constant discount factor β $(0<\beta<1)$ to be:

$$\Psi(\rho,\theta,b(L);\beta) \equiv \frac{\partial PDV(y_t)}{\partial \varepsilon_t} \equiv \sum_{i=0}^{\infty} \beta^i \psi_i(\rho,\theta,b(L)) \tag{5}$$

However, the *undiscounted, infinite* horizon effect ψ_∞ has no real economic relevance. Rather, it is used as a proxy for long horizon effects. The next proposition (also proved in the Appendix)

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[2]Exact observational equivalence would require exact correspondence between the two classes, rather than arbitrarily close approximation.

[3]This result is much stronger than the oft-discussed low power of unit root tests against alternatives with roots near unity, and is not a standard nuisance parameter problem. Any hypothesis test has low power against alternatives that are near the null, and this low power is in fact an implication of part (i) of the Proposition. But part (ii) of the Proposition shows that *all* stationary alternatives are near the unit root null; all are limits of sequences of unit root processes. To see how this differs from the standard nuisance parameter problem, recall that in the linear regression model, the error variance σ^2 is a nuisance parameter for testing hypotheses regarding the slope parameter β, e.g. $\beta=0$. Power depends on σ^2, and in fact power against any alternative β can be made arbitrarily low by choosing σ^2 sufficiently large. But the hypotheses H_0:$\{(\beta,\sigma^2):\beta=0\}$ and H_1:$\{(\beta,\sigma^2):\beta\neq 0\}$ are not nearly observationally equivalent, because there is no way to construct a sequence of processes in H_0 that converges to *any* fixed process in H_1. Near observational equivalence requires that such a sequence exist for *every* process in H_1.

shows that this use is not justified in the general case: the presence or absence of a unit root has no implications for the impulse response function at any finite horizon, nor for the discounted infinite horizon impulse response function at any positive discount rate.

Proposition 2. (i) Along the sequences defined by (2), for all finite j:

$$\begin{aligned} \lim_{i\to\infty} \psi_j(\rho_i, 0, b(L)) &= \psi_j(1, 0, b(L)) \\ \lim_{i\to\infty} \Psi(\rho_i, 0, b(L); \beta) &= \Psi(1, 0, b(L); \beta) \end{aligned} \tag{6}$$

(ii) Along sequences defined by (3), for all finite j:

$$\begin{aligned} \lim_{i\to\infty} \psi_j(1, \theta_i, b(L)) &= \psi_j(0, 0, b(L)) \\ \lim_{i\to\infty} \Psi(1, \theta_i, b(L); \beta) &= \Psi(0, 0, b(L); \beta) \end{aligned} \tag{7}$$

Any property of the impulse response function except the undiscounted infinite horizon can be produced, to an arbitrarily accurate approximation, by either a stationary or a unit root process. Even if the data could provide information on the presence or absence of a unit root, that information by itself would have no implications about the behavior of the process over any relevant horizon. Recognition of near observational equivalence therefore strengthens the conclusion of Christiano and Eichenbaum (1990), who ask, "Unit roots in GNP: do we know and do we care?" and answer, "No, and maybe not." The results in this section imply in fact that we *cannot* know and we *should not* care.

2. The Principle of Parsimony and Estimates of Persistence.

It is tempting to dismiss near observational equivalence as unimportant for practical purposes. The unit root processes that closely approximate a given stationary process for statistical purposes (Proposition 1(ii)) are those that closely approximate the finite horizon impulse response function of that stationary process (Proposition 2(ii)).[4] Why then worry about the fact that such processes cannot be distinguished?[5]

This approach is unsatisfactory for two reasons. First, it leaves unspecified the question being addressed by a unit root test. For a given purpose, nearly stationary unit root processes may well be appropriately classed together with the associated stationary process, and stationary processes with roots near one classed with the associated unit root process. But what defines the two classes? Once the literal distinction between TS and DS processes is abandoned, as required by near observational equivalence, what definition of "persistence" is put in its place, and how much of it must a process have to be classified as having a "unit root"? Are different definitions appropriate for different purposes? How do the properties of various unit root tests relate to these definitions?[6]

Second, treatment of TS and DS processes as distinct classes, combined with the standard principle of parsimony for time series specification, has discouraged nested analysis of the types of finite sample persistence about which the data can be informative. This problem is best illustrated by an example, and the remainder of this paper will concern the impulse response function aggregate output.

Using quarterly data on log U.S. per capita GNP in 1987 dollars, and an estimation period of 1948:I to 1994:II, the Schwartz Information Criterion (SIC) selects as TS and DS representations (standard errors in parentheses):

$$y_t = \underset{(0.086)}{-0.238} + \underset{(0.0001)}{0.0002}\, t + \underset{(0.068)}{1.350}\, y_{t-1} - \underset{(0.068)}{0.401}\, y_{t-2} + \varepsilon_t \tag{8}$$

$$\Delta y_t = \underset{(0.0008)}{0.0028} + \underset{(0.068)}{0.380}\, \Delta y_{t-1} + \varepsilon_t \tag{9}$$

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[4]The infinite horizon response ψ_∞ goes to zero for sequences satisfying (4). However, Faust (1994) shows how to construct a sequence of unit root processes converging to any given stationary process while maintaining arbitrary $\psi_\infty > 0$.

[5]This is the perspective taken by Campbell and Perron (1991) and Stock (1993).

[6]These issues are pursued at length in Blough (1992b).

The estimates are very similar to those obtained by Rudebusch (1993) using a somewhat shorter sample. Omitting the constant and trend terms, both specifications can be written in the form:

$$(1-\rho_2 L)(1-\rho_1 L)y_t = \varepsilon_t \tag{10}$$

where the minor autoregressive root ρ_2 is 0.442 in the TS specification (8) and 0.380 in the DS specification (9). The long-run properties of the processes are determined by the dominant root ρ_1, which is estimated to be 0.908 in the TS model and fixed at one in the DS model.

The impulse response functions of the two specifications are plotted as the solid lines in Figure 1. They differ dramatically after only a few quarters: the TS impulse response peaks in the second quarter after an innovation and has fallen to 0.5 after 12 quarters on its way to zero; the DS impulse response increases monotonically, nearing its asymptotic value of 1.6 by the time six quarters have passed. These striking differences motivate the use of unit root tests to choose between the two specifications.

Equation (10) nests the TS and DS models; given the minor root ρ_2, confidence intervals for the impulse response function ψ_i to any finite horizon i could be formed from a confidence interval for ρ_1. This approach will be pursued in the next section. However, this procedure is unsatisfactory if persistence is measured by the infinite horizon response: ψ_∞ is zero for any $\rho_1<1$, but equal to $1/(1-\rho_2)$ when $\rho_1=1$. Variation of ρ_2 does vary ψ_∞, but, as noted by Cochrane (1988) and Christiano and Eichenbaum (1990), this root is determined by the short horizon properties of the process. In practice the infinite horizon responses of the TS and DS models are not nested in the specification (10).

A nesting that allows for a continuum of values of ψ_∞ may be achieved by dropping the principle of parsimony to allow near common autoregressive and moving average factors. In the representation:

$$(1-\rho_3 L)(1-\rho_2 L)\Delta y_t = (1+\theta L)\varepsilon_t \tag{11}$$

the long-run effect of a shock is given by:

$$\psi_\infty = \frac{(1+\theta)}{(1-\rho_3)}\cdot\frac{1}{(1-\rho_2)} \tag{12}$$

If ρ_3 is set equal to the dominant root of the TS process, (11) nests the TS representation when θ=-1 and the DS representation when $\theta=-\rho_3$. As θ varies within this interval, (12) shows that ψ_∞ varies from its TS value of zero to its DS value of $1/(1-\rho_2)$. As an example, the impulse response function of the process:

$$(1-0.920L)(1-0.407L)\Delta y_t = (1-0.962L)\varepsilon_t \tag{13}$$

appears as the dashed line in Figure 1. This process is constructed so that its impulse response function matches the average of the TS and DS functions for two quarters, while having an infinite horizon response of 0.8, halfway between the TS and DS models.

How do standard methods for investigating persistence perform when applied to a process like (13)? Table 1 shows how the rejection probability of a standard unit root test varies with the value of ψ_∞, holding the short-run impulse response fixed. Each entry in the table gives the rejection probability of the Augmented Dickey-Fuller test, with lag length selected by the SIC.[7] The nominal 5 percent critical value is used.

For the TS process with long-run response of zero, the test displays its well known low power against near unit root processes, rejecting with 10 percent probability. However, as ψ_∞ rises from zero, the rejection probability *rises*, exceeding 17 percent for a long-run response of one. The rejection probability then falls back, and the test is correctly sized when the long-run response is 1.6. The size distortion of unit root tests applied to processes with long-run response near zero is well known (e.g. Schwert 1989, Blough 1992a); these results show that size distortion is not confined to such processes and in fact can be worse for processes with significant positive long-run response.

Since processes such as (13) contain near common autoregressive and moving average factors, the principle of parsimony will eliminate them; standard Box-Jenkins techniques will produce estimated processes such as (8) or (9). Figure 2 illustrates this effect. The figure summarizes the results of a Monte Carlo experiment in which 2000 replications of (13) were generated. For each replication, the Augmented Dickey-Fuller test with SIC lags was performed. When the unit root null was accepted, a parsimonious DS model was estimated. When it was

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[7]See Hall (1994) regarding use of the SIC for ADF specification.

Table 1

ADF-SIC Rejection Probabilities for Simulated GNP Processes

(2000 replications. Standard errors in parentheses.)

Infinite Horizon Response								
0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
0.107 (0.007)	0.149 (0.008)	0.165 (0.008)	0.170 (0.008)	0.168 (0.008)	0.179 (0.009)	0.142 (0.008)	0.090 (0.006)	0.052 (0.005)

Source: Simulations by author. For each entry except "1.6", the parameters of an ARIMA(2,1,1) model were chosen so as to give the indicated value of the infinite horizon response, while maintaining one- and two-period impulse responses equal to the average of those for TS and DS specifications estimated on log per capita GNP data (see text). For "1.6", an ARI(1,1) process was used with autoregressive parameter 0.375. For each replication for each process, 216 observations were produced. The first 20 observations were discarded and 10 observations were reserved for lags, leaving a sample of 186 to match the GNP data. The Augmented Dickey-Fuller test statistic was calculated, with constant and trend included and with order chosen by the SIC. Table entries are fraction of replications with the test statistic less than the nominal 5 percent critical value (-3.44). Computations were performed in GAUSS.

rejected, a parsimonious TS model was estimated. Figure 2 plots the 95 percent ranges for the two resulting groups of estimated impulse response functions. *After 12 quarters, neither band includes the true impulse response function.* Standard estimation procedure will either greatly overstate or greatly understate the long-run persistence of this process.

3. Results for nested models.

This section presents the results of two methods for estimating impulse response functions for GNP that nest the parsimonious TS and DS models. The two methods correspond to the two nested representations (10) and (11).[8]

[8]Fractionally integrated (ARFIMA) models also nest trend- and difference-stationarity. Diebold and Rudebusch (1989) present ARFIMA estimates for the impulse response function of aggregate output. Their results agree qualitatively with those presented below: confidence bands are wide.

The first method is an extension of the "root local to unity" (RLU) confidence intervals developed by Stock (1991) for the dominant root in a time series. Rewrite (10) as:

$$(1-\rho_2)(1-(1+c/T)L)y_t = \varepsilon_t \tag{14}$$

where T is the number of observations. Stock shows how to use asymptotic theory to construct a confidence interval for c from the Augmented Dickey-Fuller t-statistic. If the minor root ρ_2 is treated as fixed, the impulse response function of (14) to any horizon is monotonically related to c. Upper and lower confidence bounds for the impulse response function may be obtained using the endpoints of the confidence interval for c in (14). DS processes correspond to c=0, TS processes to c<0. The confidence interval may also include explosive roots (c>0), in which case the upper confidence bound for the impulse response function will diverge. Treating ρ_2 as fixed imparts a conservative bias to the confidence bounds.

While the RLU confidence bands so constructed can nest the TS and DS impulse responses to any finite horizon, the procedure does not produce a confidence interval for the infinite horizon response. ψ_∞ is zero for c<0, 1/(1-ρ_2) for c=0, and ∞ for c>0. This occurs because DS processes appear as a single point in (10). The near common factor nesting (11) includes a continuum of DS processes, and therefore confidence regions for its parameters imply confidence intervals for ψ_∞.

Estimation of (11) confronts the well-known problems of estimation with near common factors and near unit moving average roots. Parameter estimates have very badly behaved distributions, and standard procedures for constructing confidence regions give extremely unreliable results. Blough (1994) surveys these problems and proposes a solution. Again treating

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The ARFIMA method is not pursued here, for the following reason. An ARFIMA model that nests the parsimonious TS and DS representations is $(1-\rho_3 L)(1-\rho_2 L)(1-L)^d y_t=\varepsilon_t$, where d is the differencing parameter which may take values on the real line. The TS model (8) corresponds to ρ_3=0.908, d=0, while the DS model (10) corresponds to ρ_3=0, d=1. Formation of a joint confidence region for (ρ_3,d) is problematic, however. The points (1,0) and (0,1) are equivalent, meaning that a proper confidence region cannot be elliptical and may be disjoint. Diebold and Rudebusch avoid this problem by, in effect, setting ρ_3=0 and using d alone to determine the low frequency behavior of the process. For this reason, the parsimonious TS model is not nested in their specification.

the minor root ρ_2 as fixed, the likelihood function is evaluated at all points on a grid of values of (ρ_3,θ). A "likelihood ratio confidence region" (LRCR) for these two parameters is defined as the set of values of (ρ_3,θ) that are not rejected by a likelihood ratio test. Monte Carlo evidence indicates that the LRCR has good properties for sample sizes typical of post war quarterly data. Note that adequacy of the parsimonious DS process (9) implies that the common factor line ρ_3=-θ will be included in the LRCR. Continuity of the likelihood function implies that points near that line, e.g. with ρ_3 and $-\theta$ both near one, will be included as well.

Each accepted value of (ρ_3,θ) implies an impulse response function for $\{y_t\}$. Confidence bounds for the impulse response function at a given horizon are given by the minimum and maximum of the impulse response function of (11) at that horizon over the values of (ρ_3,θ) in the LRCR. As with the RLU, treating ρ_2 as fixed imparts a conservative bias.

Figure 3 shows the RLU and LRCR 95 percent confidence bounds for the GNP impulse response function. For both procedures, the minor root ρ_3 is fixed at 0.41, the average of its values in the parsimonious TS and DS specifications (8) and (9). The Dickey-Fuller t-statistic for the quasi-differenced process $\{y_t\text{-}0.41y_{t-1}\}$ is -2.83, giving a 95 percent confidence interval of [-17,2] for the local-to-unity parameter c using Figure 2 of Stock (1991). The implied 95 percent confidence interval for ρ_1 is [0.909, 1.011]. The LRCR bounds are generated by evaluating the likelihood function over the grid

$$\begin{aligned} \rho_3 &\in \{-0.99, -0.95, -0.90, \dots, 0.90, 0.95, 0.99\} \\ \theta &\in \{1, -0.99, -0.95, -0.90, \dots, 0.90, 0.95, 0.99, 1\} \end{aligned} \tag{15}$$

While the LRCR bounds for the impulse response function may correspond to different processes for different horizons, for the most part the upper bound corresponds to (ρ_3,θ)=(0.70,-0.65), while the lower bound corresponds to (0.85,-0.99) out to 24 quarters, and to (0.90,-1) thereafter.

The 95 percent confidence bounds are wide: there is very little information in postwar data about the univariate impulse response function of real GNP beyond a very few quarters. Even at a horizon of eight quarters, the confidence interval is roughly (0.7, 1.8); at 20 quarters, it is about (0.25, 2). The LRCR gives a 95 percent confidence interval for ψ_∞ of (0, 1.98), while the RLU indicates that an explosive root cannot be ruled out.

The differences between the RLU and the LRCR bounds can be explained by differences in specification restrictions and in finite sample performance. For the upper bound, the allowance of a moving average factor in the LRCR dominates in the early quarters, while the explosive root in the RLU dominates for longer horizons. The lower bounds differ largely because of opposite finite sample biases. The Monte Carlo experiments in Stock indicate that the confidence interval of c is rather conservative (with T=200 and c=-10, the true value is covered by a nominal 95 percent confidence interval with 91 percent probability), while a simulation of the LRCR shows that it is somewhat liberal in this region.[9]

4. Concluding remarks.

The question of whether or not there is a unit root in real GNP, or in any other macro time series, is ill-defined. The classes of unit root and stationary processes are not meaningfully distinct in the absence of *a priori* specification restrictions. If the TS / DS distinction is meant to be a proxy for the long finite horizon properties of the impulse response function, understanding would be better served by stating the properties of interest directly and developing methods of inference about them that are robust to statistical specification. The examples of this approach presented here provide explicit measurements of the uncertainty about GNP persistence.

These points have obvious implications for multivariate analysis. Nothing about Propositions 1 and 2 requires $\{y_t\}$ to be a univariate process, and the extension to cointegration is straightforward.[10] Thus, multivariate impulse response functions that are sensitive to assumed orders of integration should be interpreted carefully. Multivariate methods for nesting specifications would be very useful for dealing with these issues.

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[9]With T=186 and 1000 replications, the process ρ_2=0.85 and θ=-0.99 is covered by a nominal 95 percent LRCR with 97 percent probability.

[10]See Blough (1992b).

Appendix.

Proof of Proposition 1.

Every observation of the process (1) can be written as a function of the parameters of the process and the underlying random variables:

$$\begin{gathered} y_t = y_t(\lambda, \phi) \\ \lambda \equiv (y_s; \varepsilon_{s+1}, \ldots, \varepsilon_T) \\ \phi \equiv (\rho, \theta, b(L)) \end{gathered} \tag{A1}$$

Proposition 1 follows from continuity of the functions $y_t(.)$ in ρ and θ, for every $b(L)$ and for every realization of the random variables ϕ. To see this, use repeated substitution for y_{t-1} in (1) to find:

$$y_t = \rho^{t+s} y_s + \sum_{i=0}^{t+s-1} (\rho L)^i b^*(L) \varepsilon_t \tag{A2}$$

where

$$b^*(L) \equiv (1 - \theta L) b(L)$$

Equation (A2) does not suffice because it potentially involves an infinite number of coefficients. But since the innovations prior to period s are zero by assumption, the coefficients on those innovations are irrelevant and (A2) can be rewritten:

$$y_t = \rho^{t+s} y_s +_t b^{**}(L) \varepsilon_t \tag{A3}$$

where $b^{**}(.)$ is a polynomial of order t+s-1 defined by:

$$\begin{gathered} {}_t b_0^{**} = 1 \\ {}_t b_0^{**} = \sum_{i=0}^{j} \rho^i b_{j-i}^{**} \quad 0 < j < t + s \end{gathered}$$

Since the coefficients of $b^{**}(.)$ are continuous in ρ and θ, it follows that along sequences (3),

$$\lim_{i\to\infty} y_t(\rho_i, 0, b(L)) = y_t(1, 0, b(L)) \tag{A4}$$

for all t, for all realizations of the random variables λ, and for all $b(L)$. Similarly, along sequences (4),

$$\lim_{i\to\infty} y_t(1, \theta_i, b(L)) = y_t(1, -1, b(L)) = y_t(0, 0, b(L))$$

(A5)

and the Proposition follows immediately.

Proof of Proposition 2.

By repeated substitution,

$$\psi_j(\rho, \theta, b(L)) = \sum_{i=0}^{j} \rho^i b^*_{j-i} \tag{A6}$$

where $b^*(L)$ is defined in the proof of Proposition 1. Continuity for finite j is obvious. For the present value of an innovation, write

$$\Psi(\rho, \theta, b(L); \beta) = \sum_{j=0}^{\infty} \beta^j \sum_{i=0}^{j} \rho^i b^*_{j-i} \tag{A7}$$

which can be rearranged to obtain

$$\Psi(\rho, \theta, b(L); \beta) = \frac{b^*(\beta)}{1 - \beta\rho} \tag{A8}$$

which is continuous in ρ and θ as long as it is finite; $b(1)$ is finite by assumption, so $b^*(1)=(1-\theta)b(1)$ is finite, hence $b^*(\beta)$ is finite *a fortiori* for $0<\beta<1$.

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Figure 1

Possible GNP Impulse Response Functions

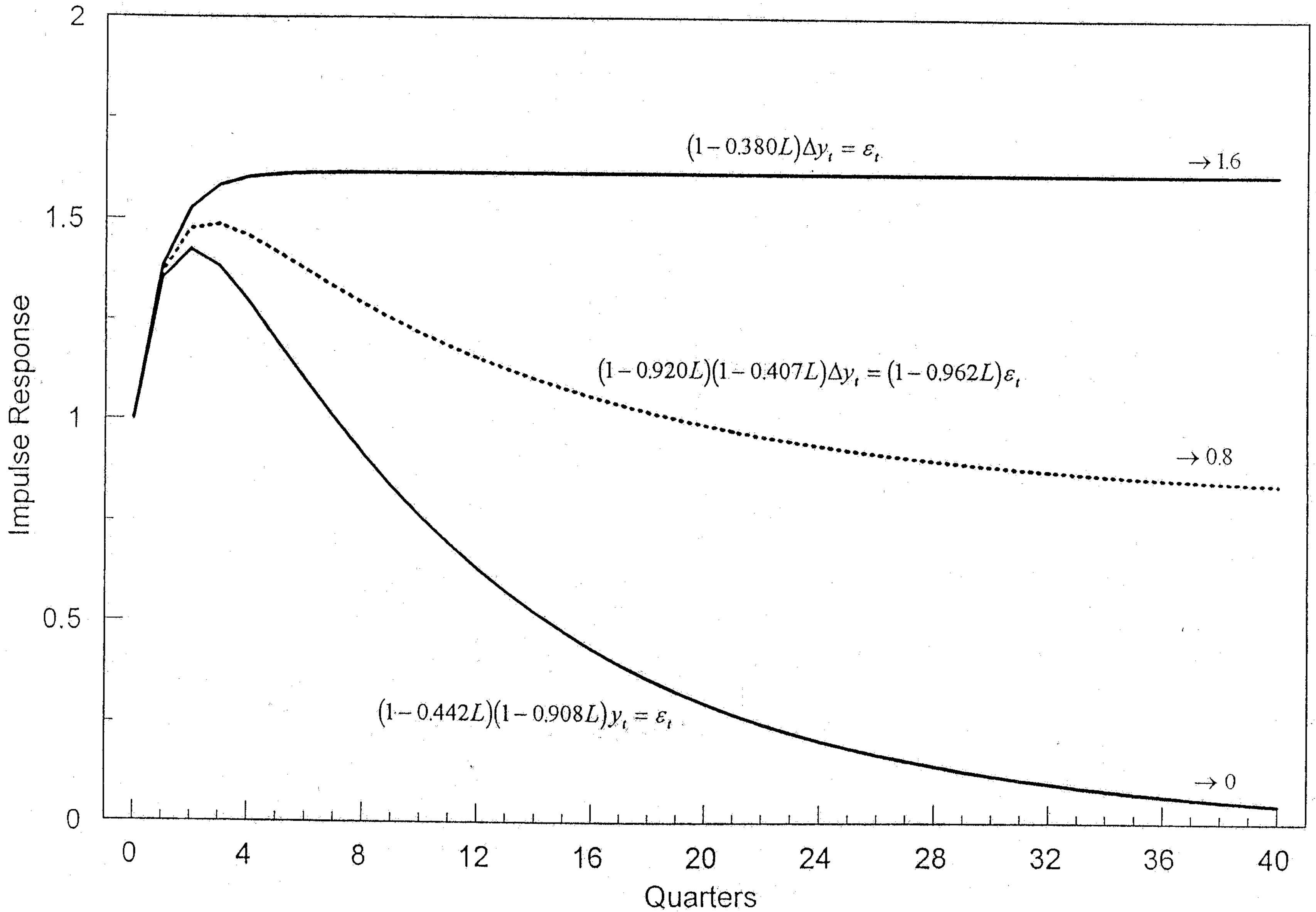

Figure 2

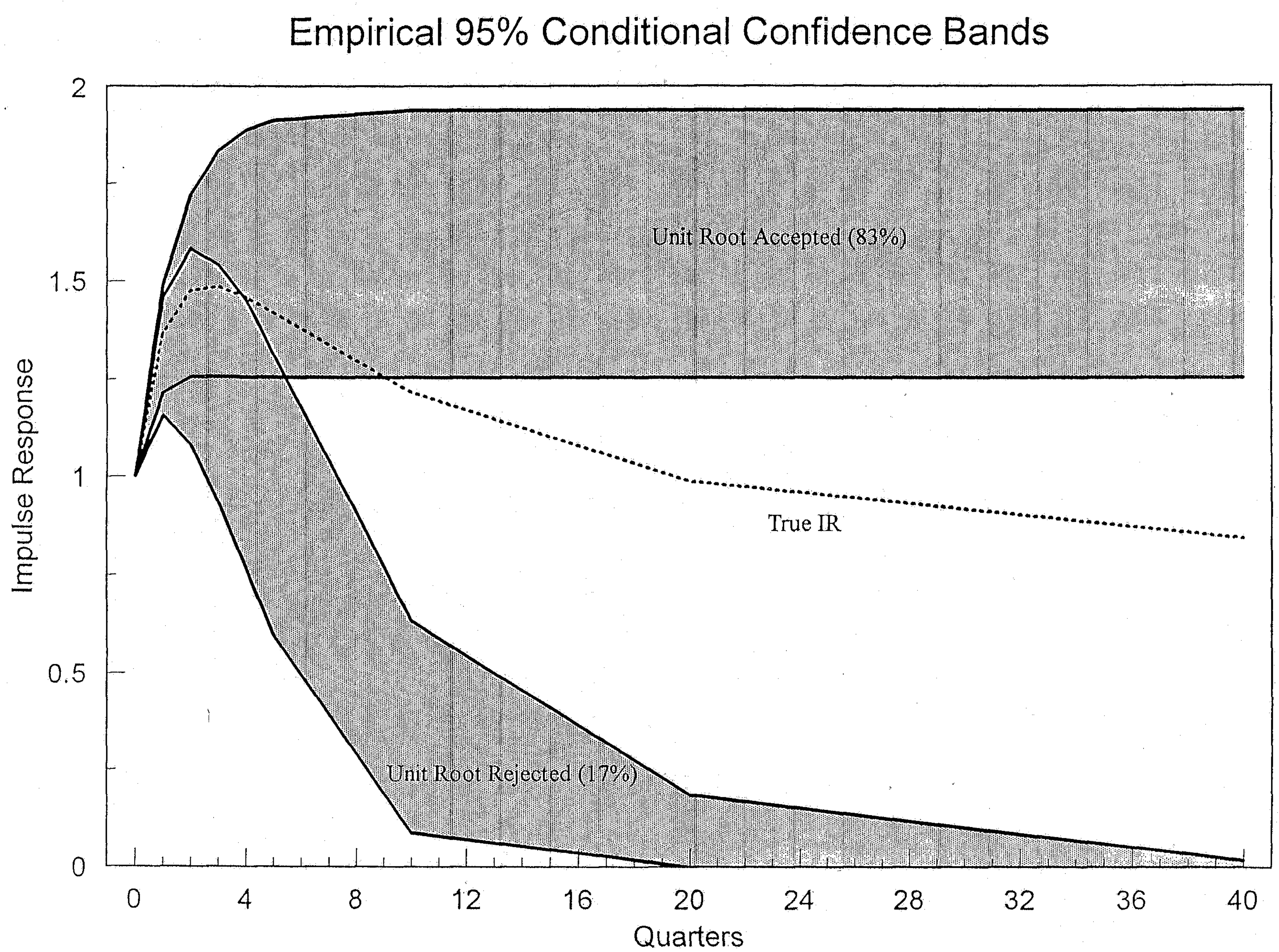
Empirical 95% Conditional Confidence Bands
Impulse Response
2
1.5
1
0.5
0
Unit Root Accepted (83%)
True IR
Unit Root Rejected (17%)
0
4
8
12
16
20
24
28
32
36
40
Quarters

Figure 3

95% Confidence Bands, GNP Impulse Response Function