Working Papers

Federal Reserve Bank of Boston

No. 04-2

# Estimating Forward-Looking Euler Equations with GMM Estimators: An Optimal Instruments Approach

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**Abstract:**
This paper compares different methods for estimating forward-looking output and inflation Euler equations and shows that weak identification can be an issue in conventional GMM estimation. The authors propose a GMM procedure that imposes the dynamic constraints implied by the forward-looking relation on the instruments set. This "optimal instruments" procedure is more reliable than conventional GMM, and it provides a robust alternative to estimating dynamic macroeconomic relations. Empirical applications of this procedure suggest only a limited role for expectational terms.

JEL Codes: C1, C15, E3

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The authors thank Sharon Kozicki for helpful comments and suggestions and Elizabeth Walat for excellent research assistance.



This paper, which may be revised, is available on the web site of the Federal Reserve Bank of Boston at http://www.bos.frb.org/economic/wp/index.htm.

**This version: July 30, 2004**

## Introduction

The basic framework for macroeconomic analysis has the structure of a simple model consisting of a demand or "IS" equation, an inflation or "AS" equation, and a monetary policy reaction function. Over the years, this model has evolved from the static Keynesian model into a micro-founded, rational-expectations model in which expectations play a prominent role in the structural equations. Expectations of current and future interest rates affect current aggregate demand, and expectations of current and future aggregate demand affect current inflation. This intertemporal specification of the basic framework, often labeled as "New Keynesian," purports to preserve the empirical wisdom embodied in the older Keynesian tradition without sacrificing the theoretical insights of modern dynamic macroeconomics.

The extent to which the New Keynesian model is able to replicate key dynamic features of aggregate data, though, remains the subject of much debate. There is a growing consensus that purely forward-looking specifications generate counterfactual dynamics for output and inflation. Some adjustment process must be added to the structural equations in order to match the inertial responses of output and inflation that are apparent in the data. While "hybrid" specifications with both forward- and backward-looking components seem better suited to characterizing actual dynamics, there is little agreement on the relative role played by expectations (that is, by the forward-looking component) in the structural equations. Different empirical studies have reached different conclusions concerning the importance of expectations of future interest rates and future demand in determining the dynamics of current output and inflation.[1]

In this study, we provide an explanation for the disparate nature of the empirical results on forward-looking demand and inflation relations. As in previous research, we document weak identification in Generalized Method of Moments (GMM) estimation of these macroeconomic relations. Weak instruments lead to GMM point estimates, hypothesis tests, and confidence

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[1] For example, Galí and Gertler (1999) argue that expectations about current and future demand pressures are the main determinant of current inflation, with past inflation playing a relatively small role. Others, for example, Fuhrer (1997), have reached a contrary conclusion that the inertial or backward-looking component, captured by past inflation, is very important in explaining current inflation.

intervals that are unreliable. In an effort to improve the small-sample properties of GMM, we propose a GMM procedure that, instead of instrumenting by means of simple linear projections on the instrument set, uses projections that impose the dynamic constraints implied by the forward-looking relation.

We label this approach to estimating forward-looking relations an "optimal" instruments approach. Any forward-looking relation can be expressed in reduced form, provided that a rational expectations solution exists. The optimal instruments approach explicitly takes into account the constraints placed on the reduced form by the posited structural relation. Conventional GMM estimation, instead, generates instruments simply by means of unconstrained linear projections on the instruments set. This difference in constructing instruments is at the root of the weak identification bias of conventional GMM estimation. In Monte Carlo simulations we show that, in contrast to conventional GMM estimation, GMM estimation with optimal instruments produces estimates that are properly centered around the true values.

The optimal instruments approach is typically used in maximum likelihood (ML) estimation of forward-looking relations. It solves for model-consistent expectations using the Anderson and Moore (1985) procedure and applies maximum likelihood to the restricted reduced form. Previous literature has shown that relative to conventional GMM estimation, ML estimation provides small-sample estimates that are less biased, more efficient, and dynamically stable. Indeed, in a weak identification context the extent to which ML dominates conventional GMM is striking.[2]

For the appropriate choice of instruments, a maximum likelihood estimator can be expressed as an equivalent instrumental variables estimator.[3] It is then not very surprising that GMM estimation with optimal instruments inherits the consistency feature of ML estimation. And, in contrast to ML estimation, GMM estimation does not require the assumption of normality of the structural shocks.[4]

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[2] See Fuhrer, Moore, and Schuh (1995) and Fuhrer and Rudebusch (2004).

[3] The instrumental variables interpretation of maximum likelihood is originally attributable to Durbin (1963).

[4] Still, to the extent that such an assumption is satisfied by the data, ML estimates will be more efficient, since maximum likelihood exploits the variance-covariance structure of the shocks.

Practitioners now have several ways of testing for the presence of weak instruments in GMM estimation. Given the variety of pathologies that GMM exhibits in a weak identification setting, applied researchers should use the available tools to assess whether weak instruments potentially are a problem in a given application. If this is the case, an optimal instruments approach provides a useful alternative to estimating macroeconomic relations with expectational terms. After all, the hallmark of these forward-looking models is precisely to impose a constrained reduced form that is the rational-expectations solution to the relation at hand.[5] The main message of the present work is that optimal instruments are sufficiently strong to center properly the distribution of estimates on the true values in a context where conventional GMM procedures exhibit weak identification.

The rest of the paper proceeds as follows. Section I describes our empirical specification. We consider an Euler equation that allows for both expectational and inertial dynamics. Such a specification has been applied – with modifications that are not crucial for the scope of our analysis – to the estimation of both demand and inflation equations in the previous literature. Euler equations for demand and inflation are isomorphic, and the issues that arise in the estimation of an Euler equation for demand and an Euler equation for inflation are, to a large extent, the same. For this reason, at this stage we consider a specification that, while stylized, is general enough to be cast into both an "IS" and an "AS" framework. While we are ultimately interested in demand and inflation relations, the Euler specification that we consider can readily be extended to many other contexts, for example, to inventory or taxation dynamics.

Section II contrasts estimation results for our Euler specification using different estimation techniques in Monte Carlo experiments. We show that the weak-identification bias present in conventional GMM estimation disappears once we use an optimal instruments approach. Estimates obtained by GMM with optimal instruments are comparable to the estimates obtained via maximum likelihood.

In Section III we bring our specification to actual data. We estimate an Euler equation for output and an Euler equation for inflation using optimal instruments and show that for both

[5] Kozicki and Tinsley (1999) provide an alternative framework for estimating forward-looking models that imposes some – but not all – restrictions implied by rational expectations.

relations the estimates indicate a larger inertial component than the one suggested by conventional GMM estimation. In other terms, conventional GMM estimates differ from optimal instruments estimates by giving expectations a more significant role. Since there is evidence that weak identification is an issue in GMM estimation of both the demand and the inflation Euler equations, we place more confidence in the estimates generated via the optimal instruments procedure. Section IV provides some concluding remarks.

## I. Model Specification

The structural relations in the New Keynesian model explicitly represent the dependence of economic decisions upon expectations regarding the future. These relations are derived from the first-order conditions (Euler equations) that characterize optimal behavior of households and firms, and they involve expectations about the future evolution of endogenous variables. The fact that the relations have microeconomic foundations is no guarantee that they are empirically realistic. Indeed, a number of authors have shown that purely forward-looking Euler equations for demand and inflation have difficulties in matching key dynamic features of aggregate data (see, for example, Estrella and Fuhrer 2002). For this reason, "hybrid" relations have been developed that depart from a purely forward-looking specification to account for the inertial responses of demand and inflation. Traditional explanations of inertia in demand and inflation rely on some form of "backwardness" in spending and price-setting decisions.

In this study, we consider a stylized hybrid Euler equation of the form:

$$z_t = (\beta - \mu) z_{t-1} + \mu \mathrm{E}_t z_{t+1} + \gamma \mathrm{E}_t x_t + \varepsilon_t , \qquad (1)$$

where $z$ and $x$ are the structural variable and the driving process in the equation, respectively, and $\varepsilon$ is a shock to the equation. The term $\mathrm{E}_t z_{t+1}$ indicates the expectation formed at time *t* of future $z$ at time $t+1$, and the term $\mathrm{E}_t x_t$, the contemporaneous expectation of $x$ (that is, we allow for the possibility that $x_t$ does not belong to the information set at time *t*). The parameter $\beta$ is generally taken to be slightly less than or equal to 1, while $\mu$ denotes the degree to which households and firms are forward-looking. When $\mu$ is equal to $\beta$, the relationship (1) becomes purely forward-looking.

When equation (1) is interpreted as a demand relation, $z$ is a measure of the output gap, $x$ is a real interest rate, and the parameter $\gamma$ enters the relationship negatively. The inertia in the output gap, captured by $z_{t-1}$, helps to explain the hump-shaped response of the output gap to policy shocks observed in VAR studies. This inertial response is usually attributed to habits in consumption expenditures (Fuhrer 2000) and to adjustment costs in the rate of investment spending (Basu and Kimball 2003). Instead, when equation (1) is interpreted as an aggregate supply relation, $z$ denotes inflation, $x$ is the output gap or another indicator of the intensity of demand in the economy, and the parameter $\gamma$ enters the relationship positively. Again, the reason for the presence of inflation inertia is largely empirical and is motivated by some form of deviation from an optimizing behavior.[6]

The crucial element in equation (1) that makes the relation intertemporal is the expectation of future $z$, $\mathrm{E}_t z_{t+1}$. This expectation enters the relation as a shifter, so that changes in $\mathrm{E}_t z_{t+1}$ shift the relation between $z_t$ and $x_t$. According to the structural relation (1), changes in $\mathrm{E}_t z_{t+1}$ are driven by changes in expectations about the future path of the driving process $x$. This can be seen explicitly by iterating equation (1) forward to obtain the following expression:

$$z_t = \zeta_2^{-1} z_{t-1} + \gamma \zeta_2^{-1} (\zeta_1 + \zeta_2) \sum_{i=0}^{\infty} \zeta_1^i \mathrm{E}_t x_{t+i} + u_t \,, \tag{2}$$

where $0 < \zeta_1 < 1$, $\zeta_2 > 1$, and $u_t$ is an error term. The parameters $\zeta_1$ and $\zeta_2$ are nonlinear functions of $\mu$ and $\beta$ in equation (1). The relation shows that $z_t$ depends on its past (the inertial or backward-looking component, $z_{t-1}$) and on the present discounted stream of $x$. In other words, current inflation is affected not only by the current output gap, but also by expectations about future output gaps. Similarly, current demand is affected by the entire term structure of (ex-ante) real interest rates. Other things equal, the larger $\mu$ in equation (1), the larger the impact of changes in expectations about the future stream of $x$ on current $z$.

[6] The deviation from the optimizing behavior can take different forms. A popular assumption is that a subset of firms set prices according to a backward-looking rule of thumb (Galí and Gertler 1999). Fuhrer and Moore (1995) appeal instead to Buiter and Jewitt's (1981) relative wage hypothesis.

## II. Investigating Estimation with Optimal Instruments

The Euler equation (1) in the previous section does not provide a closed-form solution for $z_t$. In order to obtain an expression for $z$ in reduced form, it is necessary to specify a law of motion for the driving process $x$. We assume for the moment that we can write the law of motion for $x$ and other variables affecting $x$ as follows:

$$X_t = AY_{t-1} + \eta_t, \tag{3}$$

where $X_t$ is a column vector of variables at time $t$ that includes $x$ and additional variables *other than* $z$, $Y_{t-1}$ is a column vector of lagged variables, written in first-order form, which includes all the variables in $X$ and $z$, $A$ is a matrix of coefficients, and $\eta$ a column vector of disturbances. Equation (3) describes the law of motion for $x$ and variables other than $z$ in a vector auto-regressive (VAR) form that allows for potential feedback from lagged $z$. Given the specification in (3), the reduced form for $z$ can be written as:

$$z_t = b(\mu, \beta, \gamma, A)Y_{t-1} + v_t, \tag{4}$$

where $b$ is a row vector of coefficients that depend on the parameters in equations (1) and (3), and $v$ is a disturbance term. The vector $b$ is the vector of reduced-form solution coefficients that constitute the unique, stable rational expectation solution to the Euler equation (1), given the auxiliary structure in (3).

In this context, an optimal instrument for $z$ or $x$ is an instrument that is consistent with the posited model reduced-form structure given by equations (3) and (4). Ordering $z$ first in the vector $Y$ and denoting by $B$ a matrix that vertically stacks the vector $b$ in (4) and the matrix $A$ in (3),[7] the optimal time $t-1$ instrument for $z_{t+i}$ (with $i \geq 0$) is given by:

$$\hat{z}^o_{t+i} = e_z B^{i+1} Y_{t-1}, \tag{5}$$

where the row vector $e_z$ has the first element equal to 1 and all other elements equal to zero. Similarly, the optimal time $t-1$ instrument for $x_{t+i}$ will be given by:

$$\hat{x}^o_{t+i} = e_x B^{i+1} Y_{t-1}, \tag{5'}$$

[7] That is, the matrix $B$ is written as $B = \left[\frac{b}{A}\right]$.

where now the vector $e_x$ has a value of 1 in the same position where $x$ is located in $Y$, and zero elsewhere.

The optimal instruments in (5) and (5′) impose all the constraints placed on the reduced form by the unique and stable closed-form solution to the Euler equation (1), given the auxiliary structure (3). As a result, the coefficients in $B$ are functions of the structural parameters $\mu, \beta, \gamma, A$ in (1) and (3). In contrast, conventional GMM estimation forms instruments for $z_{t+i}$ and $x_{t+i}$ simply by means of linear unconstrained projections of these variables on $Y_{t-1}$.

Note that the closed-form solution for $z_t$ relies on a specific law of motion for $x$ and any other variable that influences $x$. In other terms, a complete specification of the economic environment is needed in order to perform optimal instruments estimation. While, in principle, such a task can be daunting, it is still possible to estimate the structural relation (1) via optimal instruments using a data-consistent time-series model of the driving process. This means that the matrix $A$ in equation (3), which describes the law of motion for variables other than *z*, is left unrestricted. The coefficients in the matrix can then be estimated by simple OLS and held fixed in the estimation of (1) with optimal instruments. This general and agnostic way of modeling the driving process in a structural relation avoids the necessity of having a structural equation for each of the variables that bear on the specific relation we want to estimate. Specifying a data-consistent time-series model for the driving process also greatly reduces the risk that the estimates for the specific relation we are interested in are driven by misspecification in other relations.

### A. Estimation Methodology

Here, we briefly describe the methods used to estimate the Euler relation (1) and leave details to an appendix. The novel estimation approach to equation (1) we propose in this paper is a GMM procedure with optimal instruments. It is an iterative procedure that updates the optimal instruments at each iteration. The procedure uses an OLS estimate $A^{OLS}$ of the matrix $A$ in equation (3), with the estimate held fixed during the iteration process. The procedure starts with initial values for the parameters in (1). With these initial values and $A^{OLS}$, we compute the closed-form solution for $z$ and, using expressions (5) and (5′), optimal instruments for $z_{t+1}$ and $x_t$. The instruments are then used to estimate equation (1) via conventional GMM estimation. With the estimates of (1) and $A^{OLS}$, we compute a new closed-form solution for $z$ and new optimal instruments for $z_{t+1}$ and $x_t$. The instruments are then used to generate new estimates of (1) via conventional GMM. Such a process is repeated until the estimates in (1) converge.

The optimal instruments approach has been used in previous literature to estimate Euler relations of the form of (1) via maximum likelihood.[8] The method computes the closed-form solution for $z$ and applies maximum likelihood to the restricted reduced form (4) and the auxiliary structure (3). As with the optimal GMM procedure, it uses an OLS estimate $A^{OLS}$ of the matrix $A$ in equation (3) when computing the closed-form solution for $z$. The likelihood of the solved model can be obtained for any set of parameters under the assumption that the innovations in the model are joint normally distributed with mean zero. If the normality assumption is satisfied, ML estimation will be more efficient than GMM estimation with optimal instruments when the disturbances are correlated across equations.

We compare optimal instruments estimation with conventional GMM estimation. As already emphasized, GMM instruments for the expectational terms in equation (1) without imposing any model structure. In this context, GMM estimation is straightforward because equation (1) is linear in variables and parameters.

### B. Monte Carlo Results

In what follows, we investigate the behavior of optimal instruments estimation and conventional GMM estimation in a Monte Carlo experiment. We focus on the small-sample behavior of these different estimators under the assumption that the model is correctly specified. Several applications of GMM estimation confront what is known as “weak instruments” or “weak identification,” that is, instruments that are only weakly correlated with the included endogenous variables. When instruments are weak, the sampling distributions of

[8] See, for example, Fuhrer and Rudebusch (2004).

GMM statistics are in general non-normal; and conventional GMM point estimates, hypothesis tests, and confidence intervals are unreliable. (See, for example, Stock, Wright, and Yogo 2002.) The scope of our Monte Carlo experiment is to ascertain the extent to which optimal instruments methods improve upon conventional GMM estimation. The experiment is performed within a setting that replicates some of the relevant features that the econometrician has to confront when estimating an Euler equation for aggregate demand or inflation on actual data.

Our experiment design consists of estimating the Euler equation (1) augmented by the auxiliary structure (3) in a three-variable setup. We use three variables because the New Keynesian framework, in its simplest form, can be characterized by aggregate demand (expressed in the form of an output gap), inflation, and a short-term interest rate. Moreover, the dynamic interactions between the output gap, inflation, and short-term rates have been explored extensively in the VAR literature.

Using this three-variable setup, we perform two Monte Carlo experiments that differ in the way in which the auxiliary structure (3) is parametrized. We do so in order to cast the two experiments within an "output Euler equation" and an "inflation Euler equation" estimation framework, respectively. In the output Euler equation experiment, $z$ is the output gap and $x$ a real interest rate, while the additional variable is given by inflation. In this case, the auxiliary structure (3) consists of VAR equations for the real interest rate and inflation. In the inflation Euler equation experiment, $z$ is inflation and $x$ the output gap, while the additional variable is given by a nominal interest rate. The auxiliary structure (3) then consists of VAR equations for the output gap and the nominal interest rate.

The parameters for the auxiliary equations are estimated from actual U.S. quarterly data over the period 1966 to 2001, using three lags of each variable. The output gap is the log difference between real GDP and a measure of potential output given by the Hodrick-Prescott filter of log real GDP. Inflation is the log change in the GDP chain-weighted price index, and the nominal interest rate is the federal funds rate. The real interest rate is then given by the difference between the federal funds rate and next-period inflation.

The Euler equation (1) and the auxiliary structure (3) are used to compute 1000 replications of simulated data for a sample size of 180, with shocks drawn from a multivariate normal distribution with diagonal variance-covariance matrix, for different values of the parameters in (1). Conventional GMM estimation of equation (1) uses as instruments three lags of each of the three variables. This is the minimum lag length for the instruments that allows spanning of every realization of the endogenous variables given the assumed data-generating process.

Table 1 summarizes the results for the "output Euler equation" Monte Carlo experiment. We set the value of $\beta$ to 1.0 throughout, and estimate the parameters $\mu$ and $\gamma$ in the relation. The true value of $\gamma$ is set at 0.5 (pre-multiplied by a negative sign), and we let $\mu$ take the true values [0.1,0.25,0.5,0.75,0.9]. The top panel of the table reports summary statistics for estimates of $\mu$. The true value, $\mu^T$, is reported in the second column, followed by the mean estimate, the median estimate, and the dispersion of the estimates. The bottom panel displays the corresponding estimates of $\gamma$.

The key result that emerges from the table is that conventional GMM estimates of $\mu$ are biased. GMM overstates $\mu$ by about .15 when the true value is 0.1, and understates $\mu$ by about .12 when the true value is 0.9. In other terms, conventional GMM estimates are biased towards 0.5 from either side of 0.5. In addition, conventional GMM estimates of $\gamma$ are biased downward when the true value of $\mu$ is low. The bias disappears for high values of $\mu$ (that is, when the forward-looking component becomes more important). Estimates of $\mu$ obtained via GMM with optimal instruments, in contrast, tend to be unbiased. The table also shows that GMM estimates with optimal instruments are very close to their maximum likelihood counterparts. This is not surprising, since both techniques use the same method for constructing optimal instruments and we have posited normally distributed and uncorrelated disturbances.

Figures 1-2 illustrate the performance of conventional GMM vis-à-vis GMM with optimal instruments for various parameter values. Specifically, Figures 1a and 1b show histograms of the parameter distribution of $\mu$ and $\gamma$ when $\mu$ is equal to 0.10, while Figures 2a and 2b show histograms of $\mu$ and $\gamma$ when $\mu$ is equal to 0.90. The estimates obtained via optimal instruments are more accurate, but not more efficient, than conventional GMM estimates. It is possible to show, however, that as the sample size increases, the gain in efficiency is more pronounced for the estimates obtained via GMM with optimal instruments.

Table 2 summarizes the results for the "inflation Euler equation" Monte Carlo experiment. In the experiment we set the value of $\beta$ to 0.98 throughout and estimate the parameters $\mu$ and $\gamma$ in the relation. The true value of $\gamma$ is set at 0.10, and we let $\mu$ take the true values [0.1,0.25,0.5,0.75,0.9]. The table shows that conventional GMM estimates of $\mu$ exhibit a very pronounced bias. The GMM estimator assigns equal weight to the forward- and backward-looking components whenever the true value of $\mu$ is smaller than 0.5. That is, conventional GMM estimates of (1) do not assign a weight of more than one half to the backward-looking component on average even when such a component is preponderant. The GMM estimator is somewhat better able to recognize a specification that places greater weight on the forward-looking component (that is, $\mu$>0.5), but the estimates of $\mu$ are still biased toward 0.5. In addition, conventional GMM estimates of $\gamma$ are biased downward when the true value of $\mu$ is low, with the bias disappearing as $\mu$ increases.

When compared with conventional GMM estimates, optimal instruments estimates are more accurate. The table shows that GMM estimates with optimal instruments tend to be centered across all values taken by $\mu$. As before, estimates obtained via GMM with optimal instruments are very similar to their ML counterparts. It is important to note, however, that when the true value of $\mu$ is low, these estimates exhibit high dispersion. This is illustrated in Figure 3a, which shows histograms of the parameter distribution of $\mu$ when $\mu$ is equal to 0.10. The distribution of $\mu$ is bimodal for optimal instruments, with a pronounced mode at a value slightly less than zero. The cumulative density of $\mu$ when estimated with optimal instruments is 0.75 at $\mu$ equal to 0.40 (not shown). In contrast, the cumulative density of $\mu$ when estimated with conventional GMM is 0.10 at $\mu$ equal to 0.40 (not shown). In other words, while imprecise, the GMM estimates with optimal instruments are still better able to detect an important backward-looking component. As Figure 3b shows, the optimal instruments procedure is also better able to detect a demand effect on inflation. The precision with which $\mu$ and $\gamma$ are estimated via GMM with optimal instruments increases as the true value of $\mu$ increases. This is illustrated in Figures 4a and 4b, which show histograms of $\gamma$ when $\mu$ is equal to 0.90. In this case, optimal instruments estimates exhibit a somewhat smaller degree of dispersion than conventional GMM estimates.

Overall, the results of the two Monte Carlo exercises indicate that estimates of the Euler equation (1) obtained using optimal instruments procedures are more precise than conventional GMM methods. Maximum likelihood and GMM with optimal instruments estimates are generally unbiased and tend to behave reliably in a relevant sample size and across a range of values of $\mu$.

### C. Discussion

The Monte Carlo experiments just described rule out, by construction, model misspecification. As a result, the difference in estimates obtained by conventional GMM versus optimal instruments methods is driven entirely by finite-sample performance. For valid inference in the context of equation (1), it is necessary to have a strong set of instruments for both $\mathrm{E}_t z_{t+1}$ and $\mathrm{E}_t x_t$.

Since the number of variables to be instrumented is greater than one, simple first-stage *F*-statistics do not provide information about the joint relevance of the instruments. However, Stock and Yogo (2003) have developed a test based on Donald and Cragg's (1993) multivariate version of the *F*-statistic. Specifically, they consider a test of whether the worst-behaved linear combination of the instruments provides sufficient information about the included endogenous variables in the GMM regression. While conservative, this approach is tractable, and critical values for the test have been tabulated.

The Stock and Yogo statistic for weak instruments provides evidence that in our Monte Carlo exercises conventional GMM methods suffer from weak identification. Table 3 shows Stock and Yogo average test results for our Monte Carlo experiments. If one is willing to accept a bias as high as 20 percent of the inconsistency of ordinary least squares, then the asymptotic critical value is 6.2, and all but two of the cases in Table 3 would fail to reject the hypothesis of weak instruments in conventional GMM estimation. In contrast, GMM estimation with optimal instruments fails to provide sufficiently strong instruments only for low values of $\mu$ in the inflation Euler equation. As already mentioned, this is the instance in which optimal instruments GMM estimates exhibit a bimodal distribution away from the mean. If the acceptable bias of the instrumental variables estimator relative to ordinary least squares is 10 percent, then the critical value is 12.8, and in all cases we fail to reject weak instruments in conventional GMM estimation. Optimal instruments GMM estimation, in contrast, continues to provide strong instruments except for low values of $\mu$.

In sum, in our Monte Carlo experiments the inclusion of $z_{t-1}$ in equation (1) makes the remaining instruments too weak to produce unbiased estimates of $\mu$ and $\gamma$ under conventional GMM estimation. Instead, optimal instruments estimation methods, by imposing all the constraints placed on the reduced form by the model, provide sufficiently strong instruments to generate an unbiased distribution of estimates in most circumstances.

## III. Empirical Applications

In this section, we compare estimates for the Euler equation (1) on actual data using conventional GMM estimation and optimal instruments methods. We estimate both an output Euler equation and an inflation Euler equation. The sample period is 1966:Q1 to 2001:Q4. We use two different measures for the output gap: (i) the deviation of log real GDP from its Hodrick–Prescott (HP) filtered trend; and (ii) the deviation of log real GDP from its segmented deterministic linear trend, with breakpoints in 1974 and 1995. Inflation and interest rates are as defined in the previous section. For the inflation Euler equation, we also consider a system augmented by the inclusion of real unit labor costs in the nonfarm business sector.[9] In this four-variable system, real unit labor costs replace the output gap as the driving process in equation (1).

Conventional GMM estimation is conducted with an instruments set consisting of three lags of each of the endogenous variables plus a constant term. When performing optimal instruments estimation, the unrestricted VAR for the auxiliary structure (3) also has lag length of three. Table 4 displays estimation results for the output Euler equation. For each estimation method, the table reports two sets of estimates according to the definition of the output gap that

[9] Real unit labor costs are defined as unit labor costs in the nonfarm business sector deflated by the nonfarm business sector implicit price deflator.

is being used. Overall, lagged output appears to be an essential component across all specifications and estimation methods. The estimate of $\mu$ is one half when using conventional GMM estimation, and it is somewhat lower when optimal instruments methods are used. The real interest rate coefficient estimates are economically minute, and statistical significance is achieved in ML estimation only. Note that ML and optimal instruments GMM estimates are very close, although standard errors for the optimal instruments GMM method are larger. These results are similar to the findings of Fuhrer and Rudebusch (2003), who compare conventional GMM and ML estimates for a richer specification of the output Euler equation. Conventional GMM estimates center on a larger forward-looking component than optimal instruments estimates, but the link between output and current and future real interest rates is largely missing.

Table 5 displays estimation results for the inflation Euler equation. The table also has entries for the specification in which real unit labor costs replace the output gap as the driving process in (1).[10] In this case, the instruments set for conventional GMM estimation comprises four variables, while the vector $X$ in the auxiliary structure (3) used for the construction of optimal instruments includes real unit labor costs, the output gap, and the federal funds rate. Conventional GMM estimates are not particularly encouraging for this simple specification of the inflation Euler equation. GMM estimates suggest a larger forward-looking component than optimal instruments estimates, but the link between inflation and current and future activity, measured either by the output gap or by real unit labor costs, is either insignificant or has the wrong sign. Optimal instruments estimates have the correct sign for $\gamma$, and the "demand pressure" coefficient is significant at standard confidence levels in all instances. Again, ML estimates and GMM with optimal instruments estimates are extremely close.

Estimates in Tables 4 and 5 are suggestive of the potential differences between conventional GMM estimation and estimation with optimal instruments. While the results are specific to the simplified version of the Euler equation we have considered, it is important to

[10] In a micro-founded setup, the optimal price level is set as a markup over a present discounted value of current and future marginal costs. Thus, the driving process for inflation is better described by real marginal costs. The conditions under which real marginal costs can be well approximated by a measure of the output gap are, in fact, restrictive. See Galí and Gertler (1999).

note that in this particular context weak identification is a feature of conventional GMM estimation. It is indeed possible to show that the Stock and Yogo test statistic for instrument relevance is well below the critical value in all the conventional GMM estimates reported in Tables 4 and 5.[11] Conventional GMM provides only weak instruments for $z_{t+1}$ and $x_t$, and such a feature will continue to persist for more general specifications of (1) unless other variables explaining a higher fraction of the joint variation in $z$ and $x$ can be found.

## IV. Conclusions

Structural relations that explicitly represent the dependence of economic decisions upon expectations regarding the future provide the foundations of modern macroeconomic analysis. The degree to which agents are forward-looking has important consequences for the analysis of the character of optimal monetary policy.[12] In this context, the debate about a quantitatively realistic account of the monetary transmission mechanism remains open. Different studies have reached different conclusions about the importance of expectations of future interest rates and demand pressures on the actual dynamics of output and inflation.

This study compares different methods for estimating forward-looking output and inflation equations. Such an exercise is relevant because we suspect that the disparate nature of the extant empirical findings is largely dependent on the estimation methodology. We show that weak identification can be an issue in conventional GMM estimation of output and inflation forward-looking relations. It is thus important to resort to methods that are more reliable than GMM when instruments are weak. We propose a GMM procedure that, instead of instrumenting by means of simple linear projections on the instruments set, uses projections that impose the dynamic constraints implied by the forward-looking relation. This "optimal

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[11] Specifically, the estimated Stock and Yogo statistic is always below the appropriate critical value when the bias is no more than 20 percent of the inconsistency of OLS. This weak identification feature has already been noted by Fuhrer and Rudebusch (2004) in the context of estimating an Euler equation for ouput, and by Ma (2002) in the context of estimating an Euler equation for inflation.

[12] For example, with purely forward-looking specifications, the optimal response to an inflationary cost-push shock usually requires policymakers to allow initially for a spurt in inflation and later induce a period of deflation. In the presence of a large inertial component in inflation, it is instead optimal to bring inflation down gradually without allowing for an initial "overshooting," and endure a much larger contraction in output.

instruments" procedure is similar to maximum likelihood estimation, and offers an alternative to maximum likelihood when the assumption of normality of the structural shocks is not satisfied in the data. In contrast to conventional GMM estimation, we show that both GMM with optimal instruments and maximum likelihood provide instruments that center the parameter distributions on the true values when conventional GMM procedures exhibit weak identification.

Overall, our findings argue in favor of using optimal instruments techniques when estimating output or inflation Euler relations. Optimal instruments methods also provide a tighter test of the Euler relation because they impose a constrained reduced form that is the rational expectations solution to the relation at hand. In so doing, optimal instruments methods exploit the most distinguishing feature of dynamic rational expectations macro models.

## Appendix on GMM with Optimal Instruments Estimation Methodology

The model that comprises equation (1) and the auxiliary structure (3) can be solved using the AIM procedure of Anderson and Moore (1985) to solve for expectations of the future in terms of expectations of the present and the past. This yields the observable structure in equations (3) and (4) in the text, which we write in compact notation and first-order form as follows:

$$Y_t = B(\mu, \beta, \gamma, A)Y_{t-1} + \mathrm{v}_t .$$

The matrix $B$ is a function of the parameters in (1) and the coefficients in the auxiliary structure, while v is a vector of disturbances. The optimal instruments for $z_{t+1}$ and $x_t$ are then given by

$$\hat{z}_{t+1}^o = e_z B^2 Y_{t-1} ,$$

and

$$\hat{x}_t^o = e_x B Y_{t-1} ,$$

respectively, where $e_z$ ($e_x$) has a value of 1 in the same position where $z$ ($x$) is located in $Y$, and zero elsewhere. Denote by $Z$ the matrix containing the time series values for $[z_{t-1}, \hat{z}_{t+1}^o, x_t^o]$. Then the (unconstrained) GMM estimator of equation (1) is given by:

$$\hat{\phi} = (Z'\hat{W}Z)^{-1} Z'\hat{W}e_z Y ,$$

where $\hat{\phi}$ is a (1×3) vector, with the first element corresponding to the estimated coefficient for $z_{t-1}$, the second element corresponding to the estimated coefficient for $\mathrm{E}_t z_{t+1}$, and the third element corresponding to the estimated coefficient for $\mathrm{E}_t x_t$. The matrix $\hat{W}$ is an autocorrelation- consistent covariance estimator.

The estimation procedure starts with initial values for the parameters in (1) and uses an OLS estimate $A^{OLS}$ of the matrix $A$ in equation (3). With these initial values and $A^{OLS}$, it computes the optimal instruments $\hat{z}_{t+1}^o$ and $\hat{x}_t^o$. These instruments are then used to compute the GMM estimate $\hat{\phi}$. The estimated $\hat{\phi}$ in turn is used with $A^{OLS}$ to compute new optimal instruments and, with these new instruments, a new GMM estimate of $\phi$. Such a procedure is iterated until the estimated $\hat{\phi}$ converges.

## References

Anderson, Gary S. and George Moore. 1985. "A Linear Algebraic Procedure for Solving Linear Perfect Foresight Models." *Economics Letters*, 17, 247-252.

Basu, Susanto and Miles Kimball. 2003. "Investment Planning Costs and the Effects of Fiscal and Monetary Policy." Mimeo, Univesity of Michigan.

Buiter, Willem, and Ian Jewitt. 1981. "Staggered Wage Setting with Real Wage Relativities: Variations on a Theme of Taylor." *Manchester School of Economic and Social Studies*, 49, 3, 211-228.

Cragg, J.G. and S.G. Donald. 1993. "Testing Identifiability and Specification in Instrumental Variable Models." *Econometric Theory*, 9, 222-240.

Durbin, J. 1963. "Maximum-Likelihood Estimation of the Parameters of a System of Simultaneous Regression Equations." Paper presented at the Copenhagen Meeting of the Econometric Society.

Estrella, Arturo and Jeffrey C. Fuhrer. 2002. "Dynamic Inconsistencies: Counterfactual Implications of a Class of Rational Expectations Models." *American Economic Review*, 92, 1013-1028.

Fuhrer, Jeffrey C. 1997. "The (Un)Importance of Forward-Looking Behavior in Price Specifications." *Journal of Money, Credit, and Banking*, August, 338-350.

Fuhrer, Jeffrey C. 2000. "Habit Formation in Consumption and Its Implications for Monetary Policy Models." *American Economic Review*, 90, 367-390.

Fuhrer, Jeffrey C. and George R. Moore. 1995. *Quarterly Journal of Economics*, 110,1, February, 127-159.

Fuhrer, Jeffrey C., George R. Moore, and Scott D. Schuh. 1995. "Estimating the Linear-Quadratic Inventory Model: Maximum Likelihood versus Generalized Method of Moments." *Journal of Monetary Economics*, 35, 115-157.

Fuhrer, Jeffrey C. and Glenn D. Rudebusch. 2004. "Estimating the Euler Equation for Output." Federal Reserve Bank of Boston, working paper.

Galí, Jordi and Mark Gertler. 1999. "Inflation Dynamics: A Structural Econometric Analysis." *Journal of Monetary Economics*, 44, 195-222.

Kozicki, Sharon and Peter Tinsley, 1999. "Vector Rational Error Correction." *Journal of Economic Dynamics and Control*, 23, 1299-1327.

Ma, Adrian. 2002. "GMM Estmation of the New Phillips Curve." *Economics Letters*, 76, 411-417.

Shea, John. 1997. “Instrument Relevance in Multivariate Linear Models: A Simple Measure.” *Review of Economics and Statistics* 79, 348-352.

Stock, James H., Jonathan H. Wright and Motohiro Yogo. 2002. “A Survey of Weak Instruments and Weak Identification in Generalized Method of Moments.” *Journal of Business and Economic Statistics*, 20, 518-529.

Stock, James H. and Motohiro Yogo. 2003. “Testing for Weak Instruments in Linear IV Regression.” Harvard University, mimeo.

Woodford, Michael. 2003. *Interest and Prices*. Princeton University Press, Princeton, NJ.

# Table 1

## Properties of ML and GMM Estimators: Output Euler Equation

$$z_t = (1-\mu) z_{t-1} + \mu \mathrm{E}_t z_{t+1} - \gamma \mathrm{E}_t x_t + \varepsilon_t$$

Panel A. Estimates of $\mu$

| Estimation Method | $\mu^T$ | Mean $(\hat{\mu})$ | Median $(\hat{\mu})$ | SE $(\hat{\mu})$ |
|---|---|---|---|---|
| GMM | 0.90 | 0.79 | 0.78 | 0.14 |
| GMM | 0.75 | 0.68 | 0.68 | 0.12 |
| GMM | 0.50 | 0.48 | 0.48 | 0.09 |
| GMM | 0.25 | 0.31 | 0.30 | 0.11 |
| GMM | 0.10 | 0.25 | 0.25 | 0.15 |
| Optimal Inst. GMM | 0.90 | 0.86 | 0.86 | 0.16 |
| Optimal Inst. GMM | 0.75 | 0.72 | 0.72 | 0.13 |
| Optimal Inst. GMM | 0.50 | 0.47 | 0.47 | 0.09 |
| Optimal Inst. GMM | 0.25 | 0.24 | 0.24 | 0.11 |
| Optimal Inst. GMM | 0.10 | 0.09 | 0.10 | 0.15 |
| ML | 0.90 | 0.86 | 0.86 | 0.15 |
| ML | 0.75 | 0.71 | 0.72 | 0.13 |
| ML | 0.50 | 0.49 | 0.49 | 0.09 |
| ML | 0.25 | 0.24 | 0.24 | 0.11 |
| ML | 0.10 | 0.09 | 0.10 | 0.16 |

Panel B. Estimates of $\gamma$

| Estimation Method | $\mu^T$ | Mean $(\hat{\gamma})$ | Median $(\hat{\gamma})$ | SE $(\hat{\gamma})$ |
|---|---|---|---|---|
| GMM | 0.90 | 0.50 | 0.50 | 0.16 |
| GMM | 0.75 | 0.51 | 0.51 | 0.16 |
| GMM | 0.50 | 0.50 | 0.51 | 0.15 |
| GMM | 0.25 | 0.43 | 0.43 | 0.16 |
| GMM | 0.10 | 0.34 | 0.34 | 0.18 |
| Optimal Inst. GMM | 0.90 | 0.56 | 0.54 | 0.15 |
| Optimal Inst. GMM | 0.75 | 0.57 | 0.57 | 0.14 |
| Optimal Inst. GMM | 0.50 | 0.55 | 0.54 | 0.13 |
| Optimal Inst. GMM | 0.25 | 0.53 | 0.51 | 0.15 |
| Optimal Inst. GMM | 0.10 | 0.51 | 0.51 | 0.16 |
| ML | 0.90 | 0.55 | 0.54 | 0.16 |
| ML | 0.75 | 0.54 | 0.53 | 0.15 |
| ML | 0.50 | 0.52 | 0.53 | 0.13 |
| ML | 0.25 | 0.53 | 0.52 | 0.15 |
| ML | 0.10 | 0.53 | 0.52 | 0.17 |

Note: The true data generating process has $\mu = \mu^T$, which is displayed in the second column, and $\gamma = \gamma^T = 0.5$ in all cases.

# Table 2

## Properties of ML and GMM Estimators: Inflation Euler Equation

$$z_t = (0.98 - \mu) z_{t-1} + \mu \mathrm{E}_t z_{t+1} + \gamma \mathrm{E}_t x_t + \varepsilon_t$$

Panel A. Estimates of $\mu$

| Estimation Method | $\mu^T$ | Mean $(\hat{\mu})$ | Median $(\hat{\mu})$ | SE $(\hat{\mu})$ |
|---|---|---|---|---|
| GMM | 0.90 | 0.77 | 0.78 | 0.12 |
| GMM | 0.75 | 0.69 | 0.69 | 0.09 |
| GMM | 0.50 | 0.49 | 0.50 | 0.07 |
| GMM | 0.25 | 0.47 | 0.47 | 0.12 |
| GMM | 0.10 | 0.48 | 0.48 | 0.12 |
| Optimal Inst. GMM | 0.90 | 0.88 | 0.88 | 0.08 |
| Optimal Inst. GMM | 0.75 | 0.73 | 0.73 | 0.07 |
| Optimal Inst. GMM | 0.50 | 0.48 | 0.49 | 0.06 |
| Optimal Inst. GMM | 0.25 | 0.20 | 0.23 | 0.23 |
| Optimal Inst. GMM | 0.10 | 0.13 | 0.06 | 0.23 |
| ML | 0.90 | 0.88 | 0.88 | 0.09 |
| ML | 0.75 | 0.73 | 0.73 | 0.07 |
| ML | 0.50 | 0.48 | 0.49 | 0.06 |
| ML | 0.25 | 0.20 | 0.23 | 0.23 |
| ML | 0.10 | 0.13 | 0.06 | 0.23 |

Panel B. Estimates of $\gamma$

| Estimation Method | $\mu^T$ | Mean $(\hat{\gamma})$ | Median $(\hat{\gamma})$ | SE $(\hat{\gamma})$ |
|---|---|---|---|---|
| GMM | 0.90 | 0.11 | 0.10 | 0.04 |
| GMM | 0.75 | 0.11 | 0.11 | 0.04 |
| GMM | 0.50 | 0.11 | 0.11 | 0.06 |
| GMM | 0.25 | 0.03 | 0.03 | 0.04 |
| GMM | 0.10 | 0.02 | 0.01 | 0.03 |
| Optimal Inst. GMM | 0.90 | 0.11 | 0.10 | 0.03 |
| Optimal Inst. GMM | 0.75 | 0.11 | 0.11 | 0.03 |
| Optimal Inst. GMM | 0.50 | 0.12 | 0.11 | 0.06 |
| Optimal Inst. GMM | 0.25 | 0.12 | 0.12 | 0.07 |
| Optimal Inst. GMM | 0.10 | 0.10 | 0.09 | 0.06 |
| ML | 0.90 | 0.10 | 0.10 | 0.03 |
| ML | 0.75 | 0.10 | 0.10 | 0.03 |
| ML | 0.50 | 0.12 | 0.11 | 0.05 |
| ML | 0.25 | 0.12 | 0.12 | 0.07 |
| ML | 0.10 | 0.11 | 0.11 | 0.07 |

Note: The true data generating process has $\mu = \mu^T$, which is displayed in the second column, and $\gamma = \gamma^T = 0.1$ in all cases.

# Table 3

## Stock and Yogo (2003) Instrument Relevance Test

### A. Output Euler Equation

$$z_t = (1-\mu)z_{t-1} + \mu \mathrm{E}_t z_{t+1} - \gamma \mathrm{E}_t x_t + \varepsilon_t$$

| $\mu^T$ | GMM | Optimal Inst. GMM |
|---|---|---|
| 0.90 | 5.52 | 25.4 |
| 0.75 | 6.59 | 32.0 |
| 0.50 | 8.50 | 35.6 |
| 0.25 | 4.67 | 16.7 |
| 0.10 | 2.72 | 7.29 |

Note: The true data generating process has $\mu = \mu^T$ , which is displayed in the second column, and $\gamma = \gamma^T = 0.5$ in all cases.

### B. Inflation Euler Equation

$$z_t = (0.98-\mu)z_{t-1} + \mu \mathrm{E}_t z_{t+1} + \gamma \mathrm{E}_t x_t + \varepsilon_t$$

| $\mu^T$ | GMM | Optimal Inst. GMM |
|---|---|---|
| 0.90 | 4.56 | 16.0 |
| 0.75 | 5.40 | 19.2 |
| 0.50 | 4.54 | 14.7 |
| 0.25 | 1.32 | 1.30 |
| 0.10 | 1.29 | 1.20 |

Note: The true data generating process has $\mu = \mu^T$ , which is displayed in the second column, and $\gamma = \gamma^T = 0.1$ in all cases.

# Table 4

## Estimates of Output Euler Equation: 1966:Q1 to 2001:Q4

$$z_t = (1-\mu)z_{t-1} + \mu \mathrm{E}_t z_{t+1} - \gamma \mathrm{E}_t x_t + \varepsilon_t$$

| Estimation Method | Specification | $\mu$ | $SE(\mu)$ | $\gamma$ | $SE(\gamma)$ |
|---|---|---|---|---|---|
| GMM | HP | 0.52 | 0.053 | -0.0024 | 0.0094 |
| GMM | ST | 0.51 | 0.049 | -0.0029 | 0.0093 |
| ML | HP | 0.47 | 0.035 | 0.0056 | 0.0037 |
| ML | ST | 0.42 | 0.052 | 0.0084 | 0.0055 |
| Optimal Inst. GMM | HP | 0.47 | 0.062 | 0.0010 | 0.023 |
| Optimal Inst. GMM | ST | 0.41 | 0.064 | 0.0010 | 0.022 |

Note: The specification column provides the output trend procedure. HP is the Hodrick-Prescott filter of log real GDP, and ST is a segmented deterministic linear trend for log real GDP.

## Table 5

### Estimates of Inflation Euler Equation: 1966:Q1 to 2001:Q4

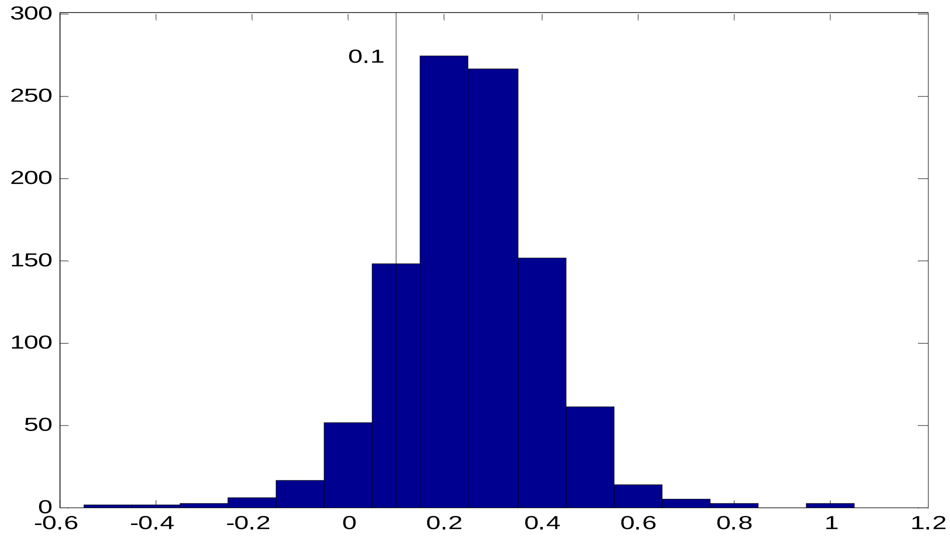
$$z_t = (1-\mu) z_{t-1} + \mu \mathrm{E}_t z_{t+1} + \gamma \mathrm{E}_t x_t + \varepsilon_t$$

| Estimation Method | Specification | $\mu$ | $SE(\mu)$ | $\gamma$ | $SE(\gamma)$ |
|---|---|---|---|---|---|
| GMM | HP | 0.66 | 0.13 | -0.055 | 0.072 |
| GMM | ST | 0.63 | 0.13 | -0.030 | 0.050 |
| GMM | *rulc* | 0.60 | 0.086 | 0.053 | 0.038 |
| ML | HP | 0.17 | 0.037 | 0.10 | 0.042 |
| ML | ST | 0.18 | 0.036 | 0.074 | 0.034 |
| ML | *rulc* | 0.47 | 0.024 | 0.050 | 0.0081 |
| Optimal Inst. GMM | HP | 0.23 | 0.093 | 0.12 | 0.042 |
| Optimal Inst. GMM | ST | 0.21 | 0.11 | 0.097 | 0.039 |
| Optimal Inst. GMM | *rulc* | 0.45 | 0.028 | 0.054 | 0.0081 |

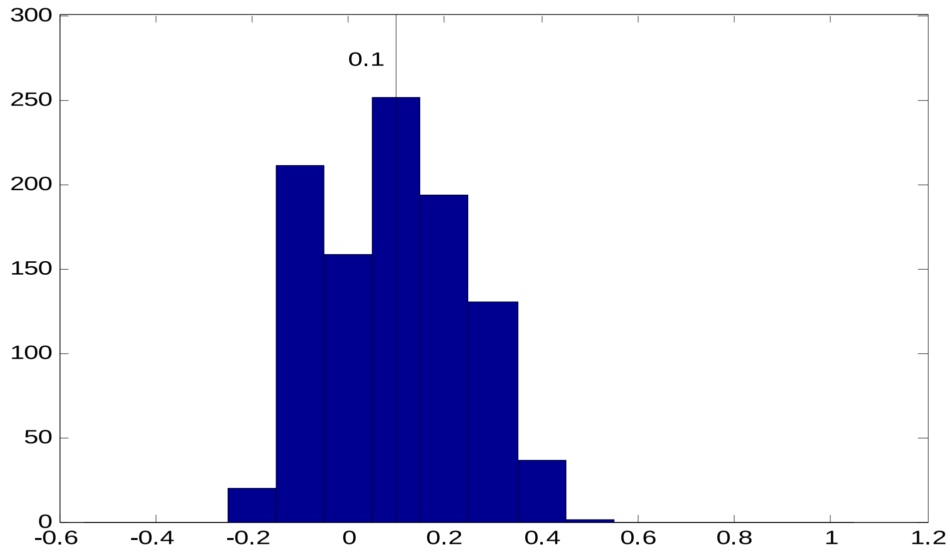
Note: The specification column provides the output trend procedure when the entry is HP or ST. HP is the Hodrick-Prescott filter of log real GDP, and ST is a segmented deterministic linear trend for log real GDP. When the entry is *rulc*, the specification replaces the output gap with real unit labor costs as the driving process in equation (1).

**Figure 1a.**

Monte Carlo Parameter Estimates of μ in Output Euler Equation.
True μ = 0.10



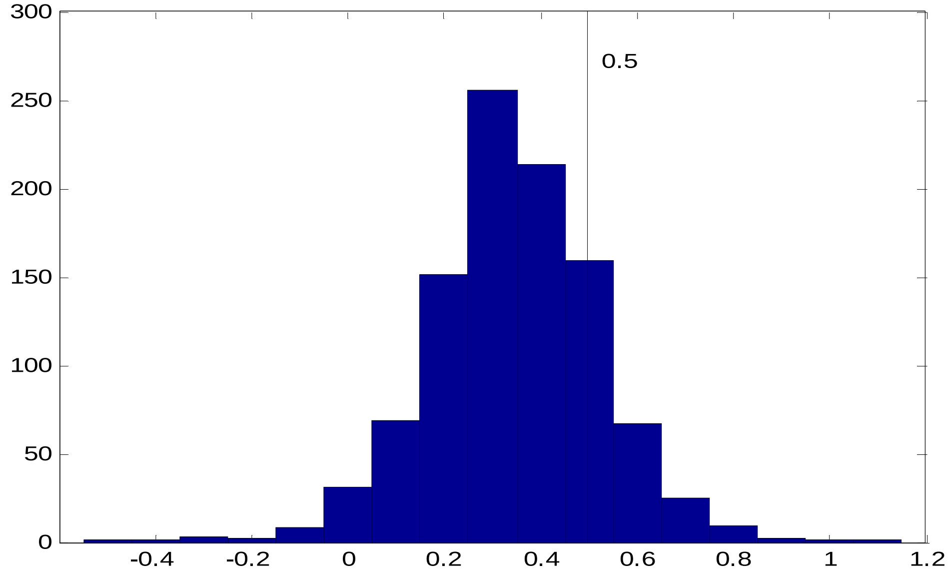

Conventional GMM Estimation




Optimal Instruments GMM Estimation

**Figure 1b.**

Monte Carlo Parameter Estimates of $\gamma$ in Output Euler Equation.
True $\gamma$ =0.50, μ = 0.10

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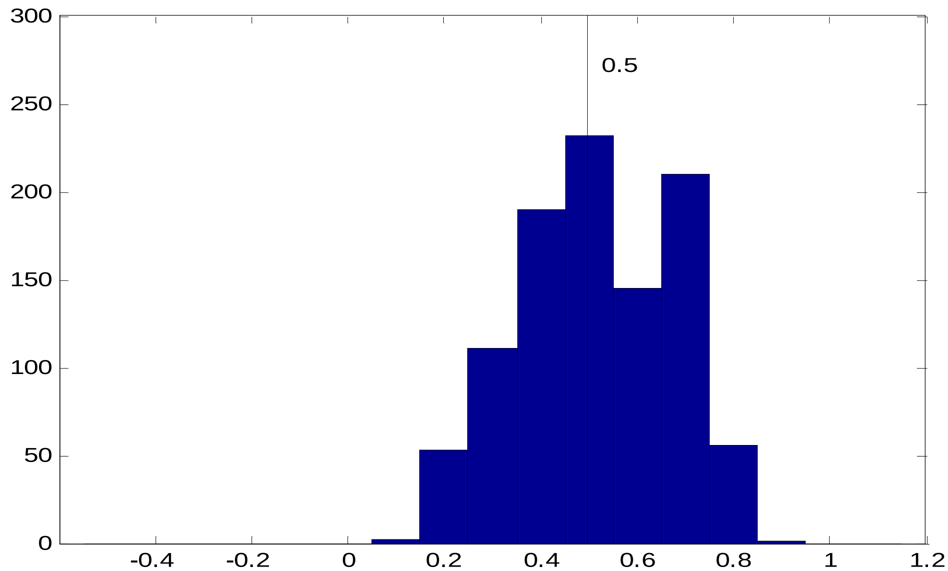
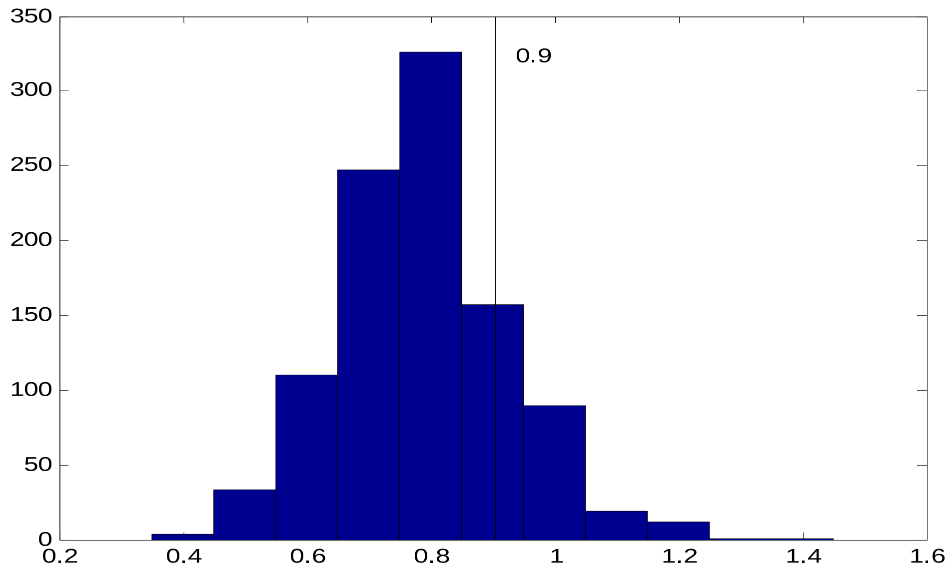



Optimal GMM Estimation

**Figure 2a.**

Monte Carlo Parameter Estimates of μ in Output Euler Equation.
True μ = 0.90

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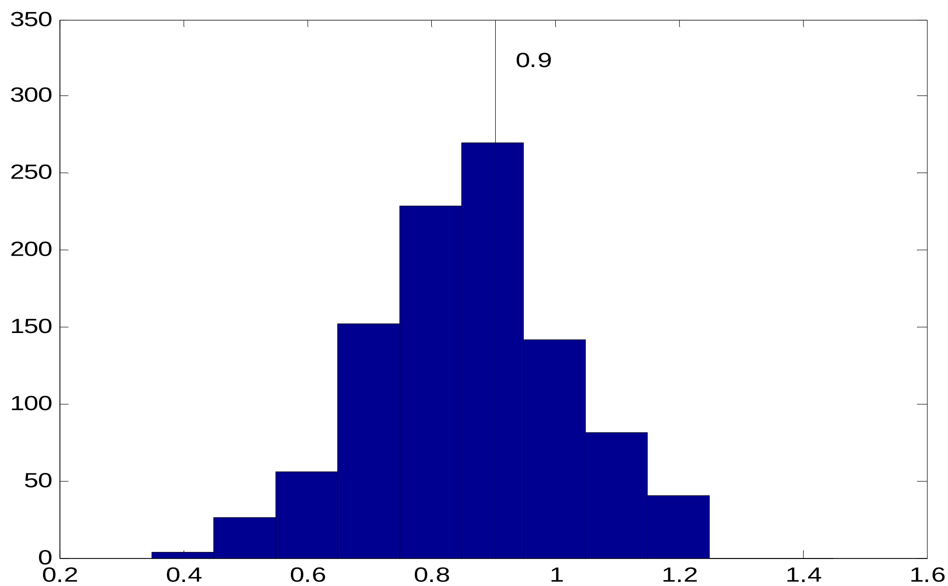
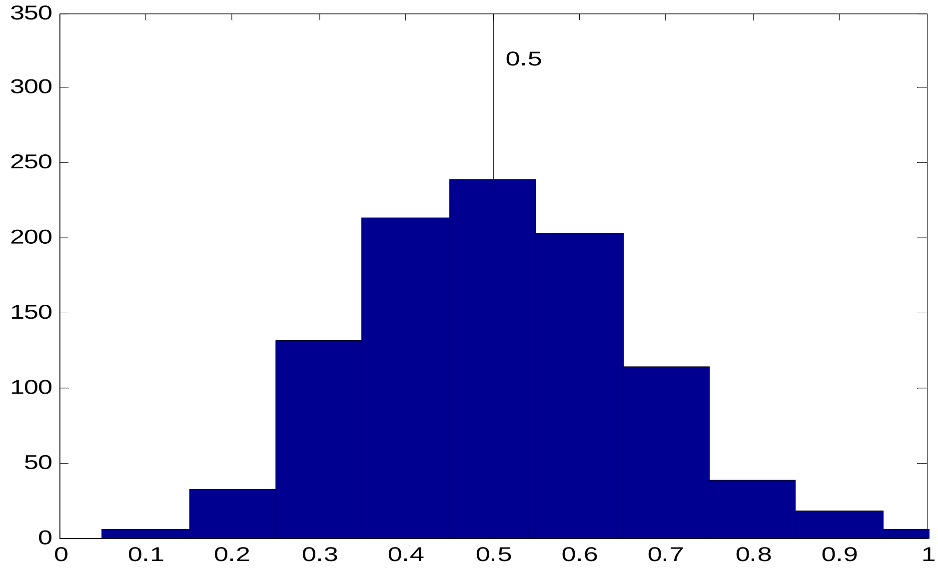


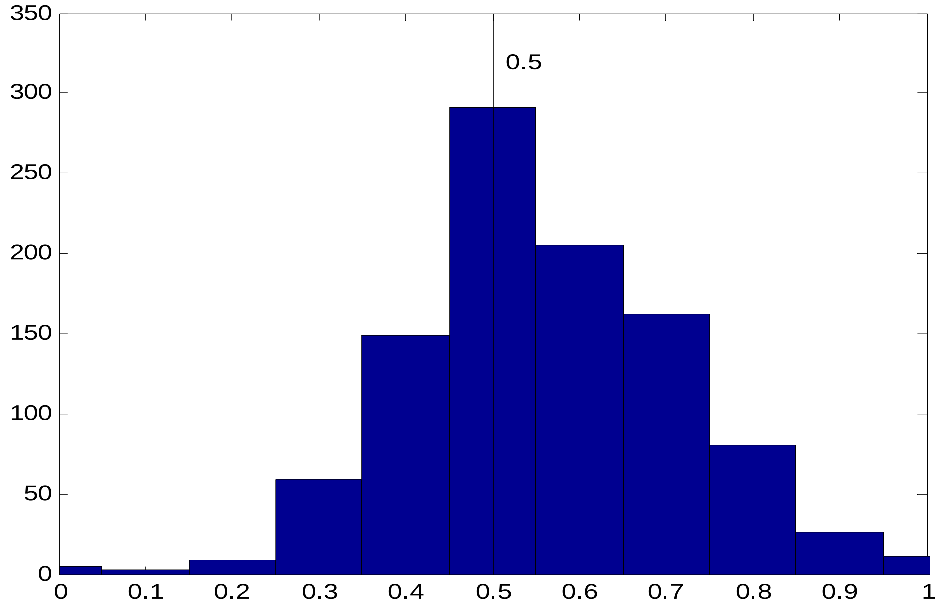

Optimal Instruments GMM Estimation

**Figure 2b.**

Monte Carlo Parameter Estimates of $\gamma$ in Output Euler Equation.
True $\gamma$ =0.50, μ = 0.90



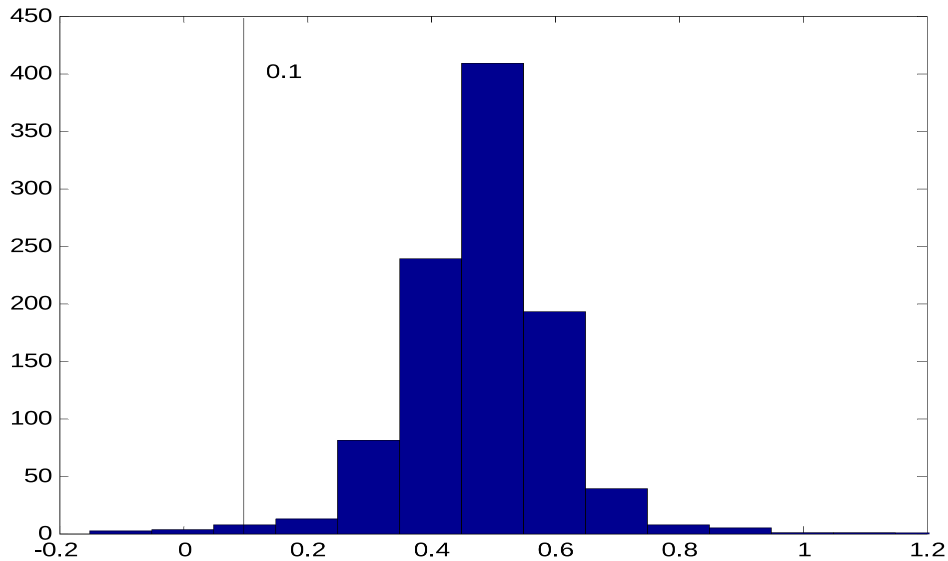

Conventional GMM Estimation



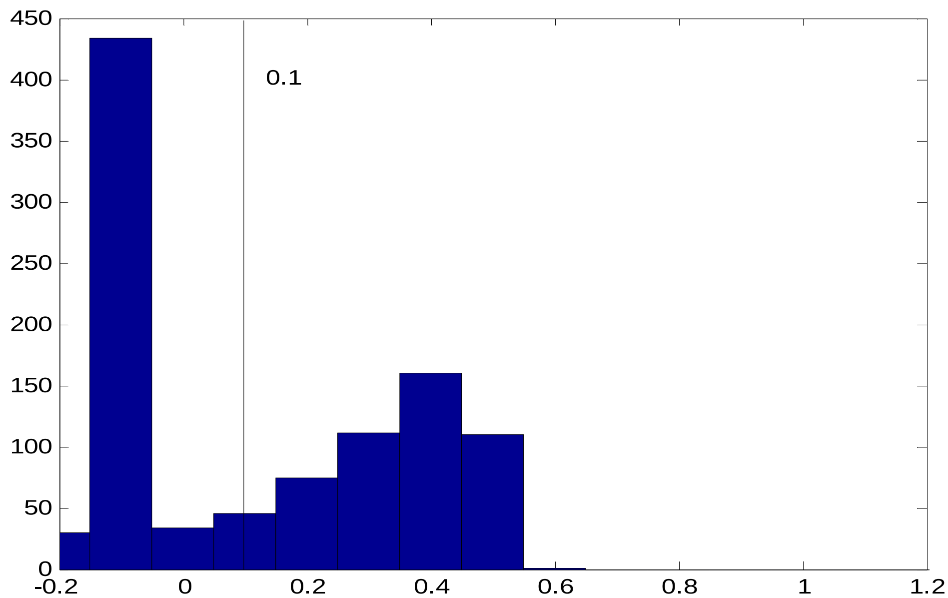

Optimal Instruments GMM Estimation

**Figure 3a.**

Monte Carlo Parameter Estimates of μ in Inflation Euler Equation.
True μ = 0.10



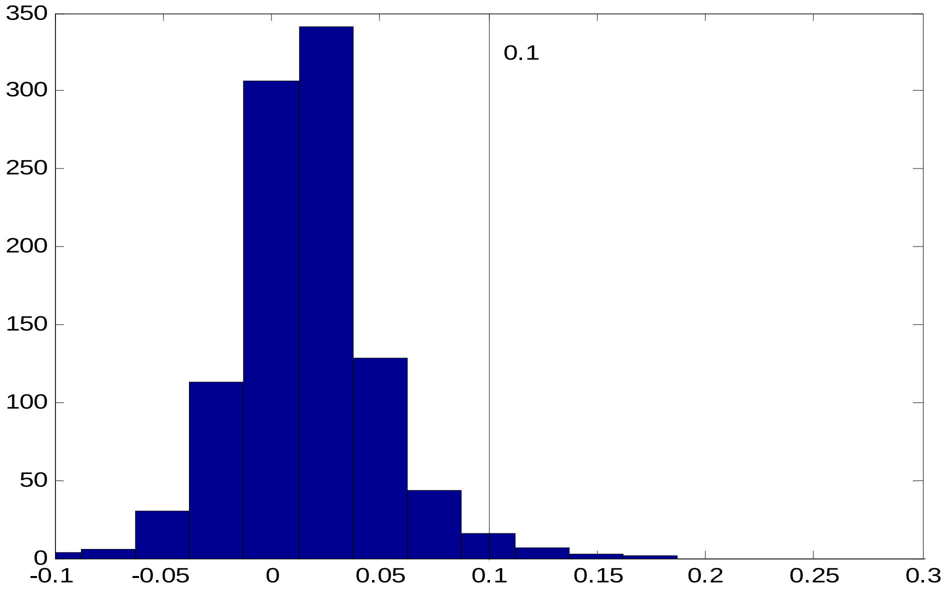

Conventional GMM Estimation



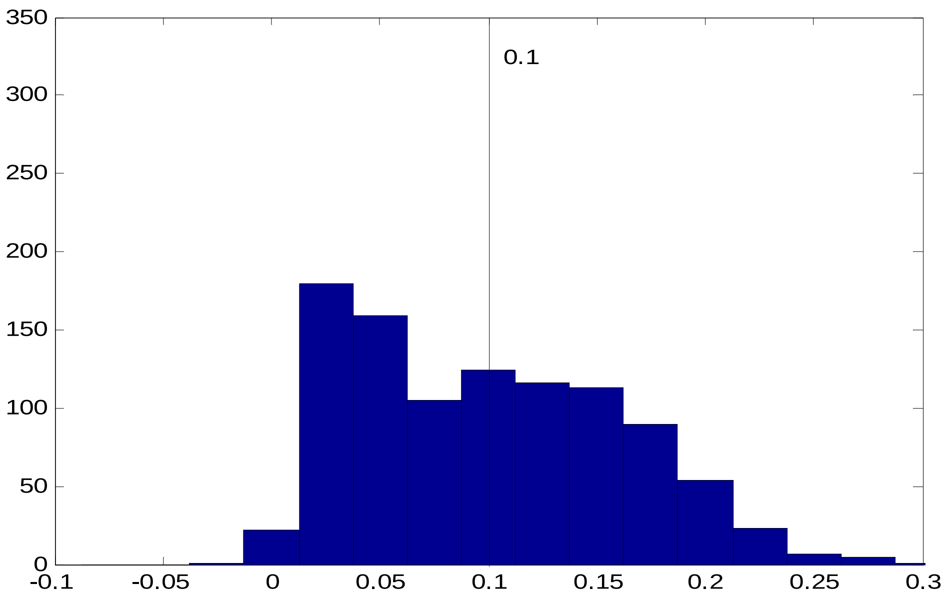

Optimal Instruments GMM Estimation

**Figure 3b.**

Monte Carlo Parameter Estimates of $\gamma$ in Inflation Euler Equation.
True $\gamma$ =0.10, μ = 0.10



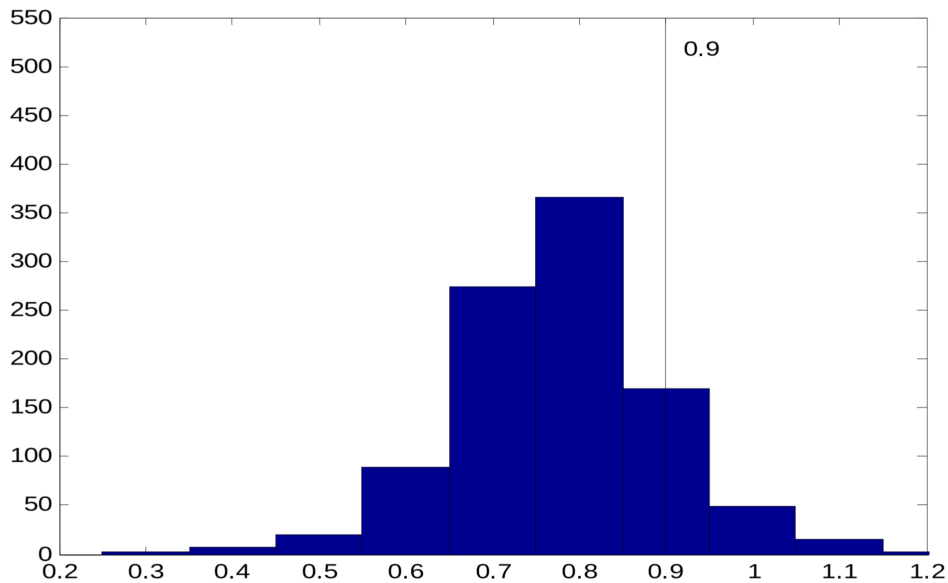

Conventional GMM Estimation



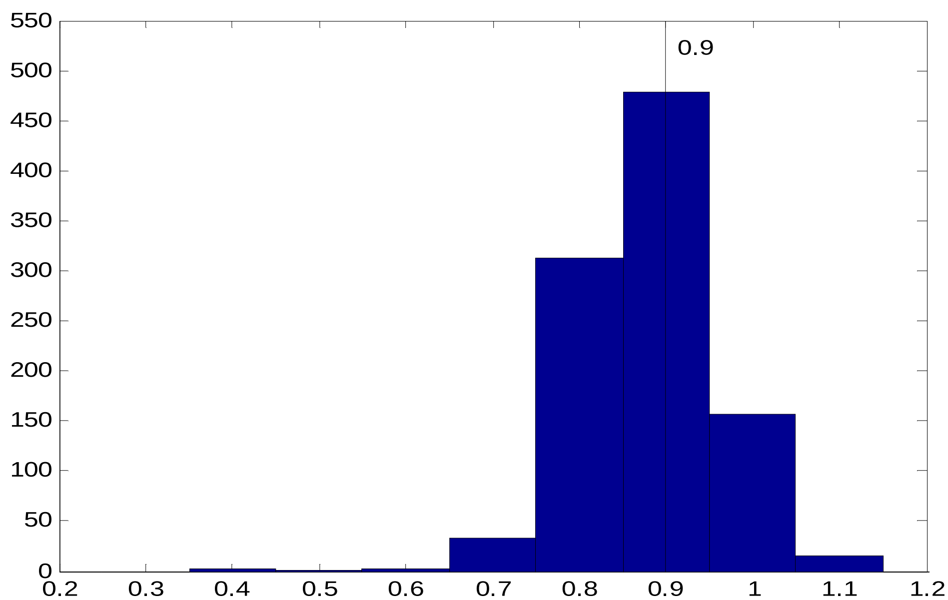

Optimal Instruments GMM Estimation

**Figure 4a.**

Monte Carlo Parameter Estimates of μ in Inflation Euler Equation.
True μ = 0.90



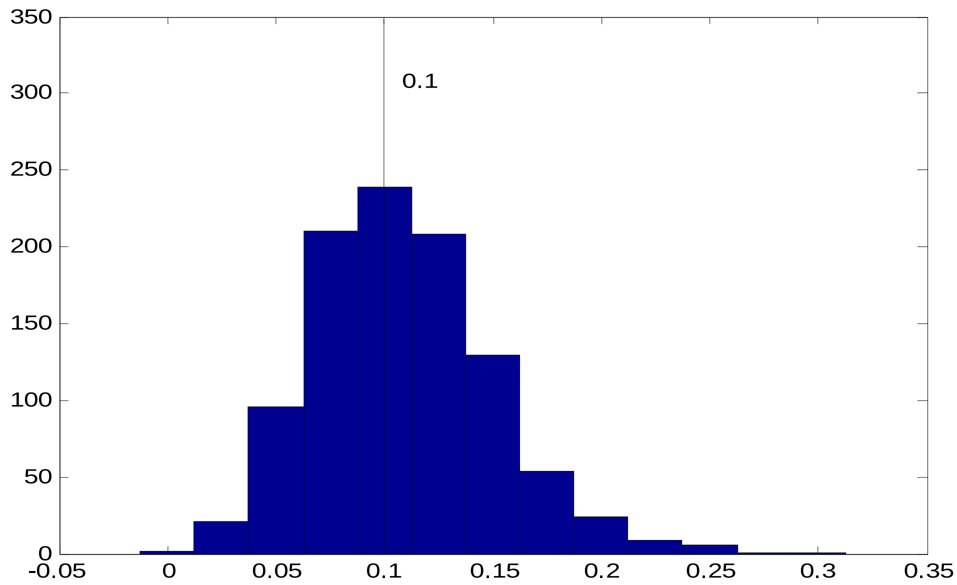

Conventional GMM Estimation



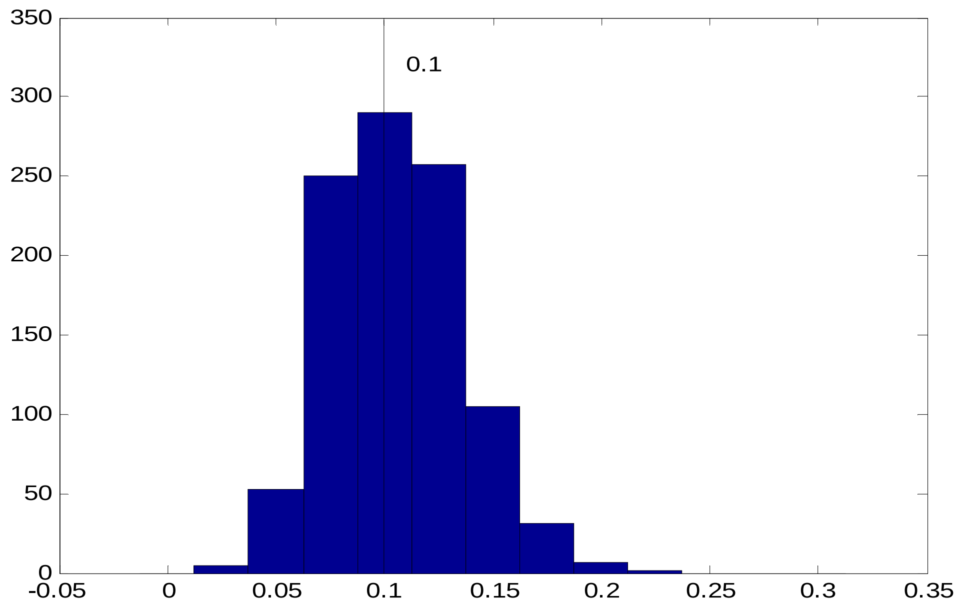

Optimal Instruments GMM Estimation

**Figure 4b.**

Monte Carlo Parameter Estimates of $\gamma$ in Inflation Euler Equation.
True $\gamma$ =0.10, μ = 0.90




Conventional GMM Estimation




Optimal Instruments GMM Estimation