Credit-Fuelled Bubbles*

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Abstract

We develop a model of credit-fuelled bubbles in which lenders accept risky assets as collateral. Asset prices and credit reinforce each other, as booming asset prices allow lenders to extend more credit, enabling investors to bid prices even higher. If investors are asymmetrically informed, there exist equilibria in which it is optimal to ride bubbles, buying overvalued assets in hopes of reselling at a profit to a *greater fool*. Lucky investors sell the bubbly asset at peak prices, only to buy it again at or below fundamental value after the crash. Unlucky investors, who buy at the peak hoping that the bubble continues to grow at least a bit longer, suffer losses. If the degree of leverage is sufficiently high, lenders repossess and liquidate the assets of unlucky investors, and may continue to seize their endowments until debts are fully repaid. In our model, raising interest rates and regulating maximal loan-to-value and loan-to-income ratios can reduce or even eliminate bubbles.

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1 Introduction

In spite of a recent surge in interest in the topic of asset price bubbles, there is a remarkable lack of consensus regarding how to model, or even define, bubbles. Economists agree that newspapers and the public at large often misuse the term bubble, using it as a synonym for boom-and-bust. Economists are of course interested in booms and busts, but they do not automatically equate price volatility with the existence of a bubble. For bubbles to exist, prices must in some sense be wrong, i.e., different from fundamental values.[1] This agreement among economists, however, turns out to be quite superficial, as different authors define fundamental value—and hence bubble—very differently. In some models, prices differ from fundamental values due to the existence of agents who are not fully rational.[2] In fully rational models, on the other hand, agents buy assets only if the benefits conferred by ownership justify the price. Those benefits may, depending on the environment, include dividends, liquidity, expectations of speculative gains, loosening of credit constraints, and others. While dividends are always considered fundamental, whether other benefits are considered fundamental or bubble varies widely. For example, in overlapping generations models à la Samuelson (1958) and Tirole (1985), a bubble is identified with the liquidity benefit provided by fiat money. In another well-known branch of the literature—see Allen and Gale (1990), Allen and Gorton (1993), Barlevy (2011)—price exceeds expected dividends because agents use borrowed funds and can shift losses to lenders if dividends are low. In this paper, we focus on the so-called speculative bubble literature, where agents are willing to pay more than the expected value of dividends because of an option value consisting of a chance to sell the asset at an inflated price to a so-called *greater fool*. Our starting point is the model of speculative bubbles developed by Doblas-Madrid (2012), which in turn builds on Abreu and Brunnermeier (2003), but includes only rational agents. In this setting, a bubble arises as an overreaction to a shock that is initially fundamental in nature. Under asymmetric information, agents continue to bid up an asset that they know is overvalued, rationally riding bubbles

[1] By wrong, we do not mean inefficient. In fact, in some strands of literature, bubbly equilibira are Pareto-superior to equilibria in which prices equal fundamental values.

[2] For example, in Abreu and Brunnermeier (2003) and Delong et al. (1990) there is a mix of rational and behavioral agents, in Harrison and Kreps (1978) and Scheinkman and Xiong (2003) agents are overconfident, and Lansing (2010) explores the bounded rationality case.

as long as the chance to earn speculative gains outweighs the risk of getting caught in the crash.[3] In this strand of the literature, rational agents find it optimal to behave like short-term traders, attempting to time the market, rather than behaving as long-term passive investors. In these models, the focus is always on bubbles that burst within finite time, and it is not necessary for bubbles to last forever—either in a deterministic steady-state or in expectation.

In Doblas-Madrid (2012), the arrival of a shock that raises an asset's expected dividends is assumed to coincide with the exogenous acceleration in the growth of agents' endowments. Rapid growth in the resources that can be invested is needed for prices to boom, but the source of that growth is not modeled. In this paper, we propose to fill this gap by introducing a self-reinforcing feedback loop between credit and asset prices as a way to endogenize endowment growth. The addition of a credit market provides a plausible answer to an open question and allows us us to analyze the effect of lending policies on speculative bubbles. A connection between bubbles and credit has been noted by numerous observers of both historical and recent episodes of booms and crises. Well-known accounts of historical episodes—for example Kindleberger and Aliber (2005) and Minsky (1986)—emphasize the role of cheap and abundant credit which fuels traders' efforts to seek speculative gains using borrowed money. Similarly, in an exhaustive historical study of asset markets in many countries, Borio and Lowe (2002) find that episodes of sustained rapid credit expansion are typically accompanied by booming stock or house prices and high levels of investment. Regarding more recent evidence, in the years prior to 2007, countries with the largest increases in household debt relative to income experienced the fastest rise in house prices over the same period (Glick and Lansing (2010)). Within the United States, house prices rose faster in areas where subprime and exotic mortgages were more prevalent (Mian and Sufi (2009), Barlevy and Fisher (2010), Pavlov and Wachter (2011)). Moreover, past house price appreciation in a given area had a significant positive impact on subsequent loan approval rates, according to Goetzmann et al. (2012). Finally, in a comprehensive report, the U.S. Financial Crisis Inquiry Commission (2011) emphasized the effects of a self-reinforcing feedback loop in which an influx of new homebuyers with access to easy mortgage credit helped fuel an excessive run-up in house prices, thus encouraging lenders to ease credit further on the assumption that

[3] Other rational models of speculative bubbles include Allen et al. (1993) and Conlon (2004). There is also a 'greater fool' component to Allen and Gorton (1993), in addition to the risk shifting mentioned above.

house price appreciation would continue indefinitely. While most of these references refer to the housing market, which has attracted much attention due to its macroeconomic importance, the logic of a self-reinforcing feedback loop between prices, collateral values, and credit also applies to many other financial markets. For instance, margin trading is common in equity markets, spot foreign exchange markets, futures markets, etc.

Motivated by all these observations, we extend the rational model of speculation developed by Doblas-Madrid (2012) by allowing agents to ride bubbles using borrowed money. Our model of the credit market is very stylized. Lenders costlessly and competitively intermediate between the model economy and the rest of the world. They are willing to lend at the exogenous risk-free rate, but the amount an agent can borrow is subject to a collateral constraint similar to Kiyotaki and Moore (1997). Although agents speculate with borrowed money, our setup differs from the leveraged-bubble models of Allen and Gale (2000) and Barlevy (2011), where limited liability on the part of borrowers induces them to pay more than fundamental value. The key difference is that, in our model, lenders have full recourse. That is, if the posted collateral is insufficient to pay repay the debt, they seize agents' endowments until they recover principal and interest on their loans. Therefore, we would like to emphasize that, in our model, agents' willingness to pay more than fundamental value for the risky asset is purely due to the prospect of earning speculative profits, and not due to the fact that they shift losses to third parties. Our assumption that endowments can be seized ex post, but only assets can be pledged ex-ante precisely allows speculators to borrow growing amounts in order to fuel the bubble without—for the parameter values of interest—allowing them to shift losses to lenders. Moreover, there are many settings in which our assumptions do not seem far-fetched in light of credit market regulations. While some US states allow underwater borrowers to 'walk away' from mortgages, in others, lenders sue underwater borrowers for any unpaid balance if the revenue from liquidating a foreclosed property does not satisfy the outstanding debt. In fact, the limited liability feature of mortgages in some US states is quite unusual by international standards. In most other countries—including some that have experienced dramatic housing booms and busts, such as Spain and Sweden—mortgage regulations conform to the full recourse rather than the limited liability model. Moreover, under full recourse, borrowing limits are primarily based on appraised property values, with income requirements being typically verified for approval. Full recourse regulations are also common in other financial markets besides housing. For instance, in the context of stock trading, margin agreements do not typically restrict losses to the value of collateral. The maximum position a stock investor can open depends only on the value of the collateral, i.e., on the margin deposited. However, if losses exceed the value of collateral, the investor is still liable for the outstanding balance.[4]

While the addition of a credit market does add complexity to our model, the model is still tractable enough for us to obtain a number of analytical results, at least in some regions of the parameter space. Importantly, the self-reinforcing feedback loop between prices and credit converges, after a number of periods, to a constant growth rate of the price. This simplifies the analysis and allows us to apply results from Doblas-Madrid (2012), subject to only minor modifications. Bubble duration depends on the growth rate of the bubble, which in our model is endogenous and depends positively on the maximum loan-to-value ratio allowed by the lender. Bubbles randomly generate winners and losers. Lucky agents sell at peak prices and re-purchase the asset after the crash. Unlucky agents are often forced to liquidate their assets after the crash. Depending on parameters, it is possible for the price to overshoot during the crash. This occurs when lucky agents do not have enough funds to acquire all the shares of the risky asset that are being sold in forced liquidations at fundamental value. In this case, lucky agents win twice, as the risky asset's price recovers. The model predicts that looser lending standards (as measured by higher loan-to-value ratios) imply faster growth in the price of the risky asset and therefore longer bubbles. Bubbles can also be reduced or even eliminated, by raising the risk-free interest rate or by enforcing lending caps based on price-to-income ratios.

The rest of the paper is organized as follows. Section 2 presents the model. In Section 3, we define equilibrium and propose a candidate strategy profile. In Section 4, we characterize equilibrium bubble duration in the most tractable, benchmark case. In Section 5, we discuss our results and how to extend the analysis to less-tractable cases. In Section 6, we conclude.

[4]In normal times, it is rare for a stock investor to be left with negative equity after her portfolio is liquidated. However, in so-called *fast markets,* prices move so rapidly that the equity in an investor's account may be negative by the time the broker finishes closing the positions in an under-margined account. For example, after the dot-com crash in the early 2000s, some investors still owed money on their margin loans even after their brokerage firms liquidated their entire portfolios.

2 The Model

Time is discrete and infinite with periods labeled $t = ..., 0, 1, ...$ There is a unit-measure continuum of investors indexed by $i \in [0, 1]$, and a continuum of competitive lenders. In each period, investor i earns labor income y_t, trades in the asset market, refinances debt and consumes. There are two assets, a risk-free and a risky one. The former yields an exogenous gross return R and can be turned into consumption at a one-to-one rate. The risky asset, which exists in unit supply, trades at price p_t and pays nonnegative dividends $\{d_t\}_{t\in\mathbb{Z}}$.

Investors are risk neutral, but may be hit by preference shocks inducing an urgent need to consume. Specifically, every period a randomly chosen fraction $\theta \in [0, 1]$ of investors are hit by a shock that sets their discount factor $\delta_{i,t}$ equal to 0, while the remaining mass $1 - \theta$ have discount factor $\delta_{i,t} = 1/R$. For an investor with $\delta_{i,t} = 1/R$, expected utility is given by

$$E_{i,t}\left[U\left(\{c_{i,\tau}\}_{\tau\geq t}\right)\right] = E\left[c_{i,t} + \frac{1}{R}\sum_{\tau\geq t+1}\left(\frac{1-\theta}{R}\right)^{\tau-(t+1)} c_{i,\tau}\,\middle|\, I_{i,t}\right], \tag{1}$$

where $c_{i,t}$ denotes investor i's time-t consumption, U denotes utility, and $E_{i,t}$ expectation conditional on information $I_{i,t}$. If $\delta_{i,t} = 0$, (1) reduces to $E_{i,t}c_{i,t}$. Preference shocks do not represent the investor's death. After being hit, investors continue to receive endowments and can be hit again. For simplicity, shocks are assumed i.i.d., so that the probability that $\delta_{i,t} = 0$ is independent of past values $\delta_{i,s}$, for $s < t$. This means that investors recently hit by the shock are just as likely to be hit again as investors who have not been hit for a long time.

Depending on whether she is in compliance with the terms of her outstanding loans, investor i can start period t in good credit standing, denoted by $\varsigma_{i,t} = 1$, or in default, in which case $\varsigma_{i,t} = 0$.

2.1 Investors in Good Standing

Consider an investor i starting period t with good credit $\varsigma_{i,t} = 1$, $h_{i,t} \geq 0$ units of the risky asset, $b_{i,t} \geq 0$ units of the risk-free asset, and a liability of $l_{i,t} \geq 0$ units of the risk-free asset. For future reference, the corresponding economy-wide aggregates are given by H_t, B_t and L_t, respectively. At the beginning of the period, investor i learns the realization of her preference shock and receives an endowment y_t. She then visits an asset market, modeled as a Shapley-Shubik trading post with two-step trading. In Step 1, investors submit orders to buy/sell as follows. In one bin, investor i deposits risky-asset shares $s_{i,t} \geq 0$ she offers for sale. Since the risky asset cannot be shorted, it must be that

$$s_{i,t} \leq h_{i,t}. \tag{2}$$

In another bin, she deposits $m_{i,t} \geq 0$ units of the risk-free asset she wishes to spend buying the risky asset. The risk-free asset can be shorted by borrowing from lenders. Thus, $m_{i,t}$ is bounded by

$$m_{i,t} \leq b_{i,t} + y_t, \tag{3}$$

where some or all of $b_{i,t}$ may be borrowed. Note that investors choose $m_{i,t}$ and $s_{i,t}$ before knowing the price, determined in Step 2 once all bids and offers are combined. Specifically, the ratio of the aggregate risk-free asset bid M_t over the aggregate number of shares sold S_t determines the price

$$p_t = \frac{M_t}{S_t}. \tag{4}$$

Since there is always positive mass of shock-induced sellers, $S_t > 0$ and p_t is well defined. After trading, investor i's portfolio is given by

$$h_{i,t+1} = h_{i,t} + \frac{m_{i,t}}{p_t} - s_{i,t} \tag{5}$$

and

$$\widetilde{b}_{i,t} = b_{i,t} + y_t + p_t s_{i,t} - m_{i,t}, \tag{6}$$

where $\widetilde{b}_{i,t}$ denotes interim—post asset market and pre-refinancing—risk-free asset balance.

After the asset market closes, investor i refinances her debt with a lender. We assume that interest accrues between the time the asset market closes and the debt refinancing time. In other words, by the debt refinancing stage, risk-free assets and liabilities $\widetilde{b}_{i,t}$ and $l_{i,t}$ have respecitively increased to $R\widetilde{b}_{i,t}$ and $Rl_{i,t}$. Lenders costlessly and competitively intermediate between the model economy and the rest of the world, where they borrow or lend at the rate R. To retain good credit, investor i must repay $Rl_{i,t}$. She can borrow new debt $l_{i,t+1}$, but only up to a fraction $\phi \in (0, 1]$ of the value of her risky asset shares. That is,

$$l_{i,t+1} \leq \phi p_t h_{i,t+1}. \tag{7}$$

If investor i can repay her debt, i.e., if

$$R\widetilde{b}_{i,t} + \phi p_t h_{i,t+1} \geq Rl_{i,t}, \tag{8}$$

we assume that she repays and maintains good credit. Thus, we assume that lenders are able to monitor assets in order to preclude borrowers from diverting borrowed funds to increase consumption.[5] We also assume, without loss of generality, that investors remaining in good credit standing borrow as much as they can, so that (7) holds with equality. Since the interest rate on risk-free assets and liabilities is the same, an investor who borrows at t and passively keeps the funds in the form of risk-free balances next period can always repay her loan and incurs no cost.[6] Finally, investor i splits the remaining $R\left(\widetilde{b}_{i,t} - l_{i,t}\right) + l_{i,t+1}$ beween consumption and savings. That is, she consumes a fraction $\xi_{i,t} \in [0, 1]$ of the available balance

$$c_{i,t} = \xi_{i,t}\left[R\left(\widetilde{b}_{i,t} - l_{i,t}\right) + l_{i,t+1}\right]. \tag{9}$$

and saves the rest

$$b_{i,t+1} = \left(1 - \xi_{i,t}\right)\left[R\left(\widetilde{b}_{i,t} - l_{i,t}\right) + l_{i,t+1}\right]. \tag{10}$$

Consumption choices are also straightforward. If $\delta_{i,t} = 0$, investor i chooses $\xi_{i,t} = 1$, consuming everything and saving nothing. And if $\delta_{i,t} = 1/R$, we assume that she chooses the opposite

[5]Moreover, note that investors hit by the preference shock do not borrow. Since $\phi \leq 1$, they consume more by selling all risky shares than by borrowing against them.

[6]It would be more realistic to assume that agents do not borrow more than the maximum they expect to bid in the next period. However, incorporating this would greatly complicate the exposition—requiring us to set up two, instead of one, decision problems per period—without affecting results.

$\xi_{i,t} = 0$, consuming nothing and saving everything.[7]

2.2 Investors in Default

If (8) is violated, investor i defaults and $\varsigma_{i,t+1}$ becomes 0. The lender seizes $R\widetilde{b}_{i,t}$ and $h_{i,t+1}$, immediately applying $R\widetilde{b}_{i,t}$ towards debt repayment, and slating $h_{i,t+1}$ for liquidation as soon as the asset market reopens in period $t+1$. Investor i is given no allowance for consumption, and thus $c_{i,t} = b_{i,t+1} = 0$ and $l_{i,t+1} = R\left(l_{i,t} - \widetilde{b}_{i,t}\right)$.

As long as she remains in default, investor i is not free to make any choices. When the asset market opens at time $t+1$, she is forced to set $(m_{i,t+1}, s_{i,t+1}) = (0, h_{i,t+1})$. That is, she must sell her shares of the risky asset—which may be positive only in the first period of default—and she must keep her endowment in the form of risk-free balances, i.e., she is not allowed to invest it in the risky asset. Thus, she leaves the asset market with $(\widetilde{b}_{i,t+1}, h_{i,t+2}) = (y_t + p_t h_{i,t}, 0)$.

At the lender's, $R\widetilde{b}_{i,t+1}$ is applied towards debt repayment. If this does not suffice to repay the outstanding debt, investor i remains in default and must consume zero. If it suffices, she regains good credit and the freedom to make choices, starting with the decision to consume or save the balance left over after repayment $R\left(\widetilde{b}_{i,t+1} - l_{i,t+1}\right)$. In sum,

$$\begin{pmatrix} \varsigma_{i,t+2} \\ l_{i,t+2} \\ c_{i,t+1} \\ b_{i,t+2} \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ R\left(l_{i,t+1} - \widetilde{b}_{i,t+1}\right) \\ 0 \\ 0 \end{pmatrix} & \text{if } l_{i,t+1} > \widetilde{b}_{i,t+1} \\ \begin{pmatrix} 1 \\ 0 \\ \xi_{i,t}\left[R\left(\widetilde{b}_{i,t} - l_{i,t}\right) + l_{i,t+1}\right] \\ \left(1-\xi_{i,t}\right)\left[R\left(\widetilde{b}_{i,t} - l_{i,t}\right) + l_{i,t+1}\right] \end{pmatrix} & \text{if } l_{i,t+1} \leq \widetilde{b}_{i,t+1}. \end{cases} \quad (11)$$

Endowments are assumed valuable enough (relative to the maximum possible decline in asset prices) to repay any debt.[8] In other words, the number of periods that investor i remains in

[7] If the risky asset is expected to yield more than the inverse of the discount factor, consuming zero is the only optimal choice. On the other hand, if expected returns equal the inverse of the discount factor, any amount is optimal. Then, we assume that zero consumption as a tie-breaking rule.

[8] A parameter restriction ensuring that this is the case is derived later in the paper.

Figure 1: [Insert here]

default is always finite. As discussed in the introduction, this allows us to consider very stylized lenders who are willing to lend at the risk-free rate, since in the long run they do not incur any credit losses.

2.3 Timeline: Three Phases

As in Doblas-Madrid (2012), there are three phases: A pre-boom phase for $t < 0$, a boom from time 0 to the (endogenous) crash period t_c, and a final, post-crash phase. As we will see later, in the equilibria of interest the boom will be subdivided into a fundamental part and a bubble. The full timeline is shown in Figure 1.

In the pre-boom phase, the risky asset's fundamental value is known by all and equal to the expected value of future dividends. This value, discounted to time 0, equals $\overline{F} > 0$. Assuming, for simplicity, that all dividends are expected to be paid after date 0, the fundamental value for any $t < 0$ is given by

$$f_t = \overline{F}R^t. \tag{12}$$

Thus, while $t < 0$, the fundamental value f_t of the risky asset and its price p_t are both equal and given by $\overline{F}R^t$. In this preliminary phase, the risky asset is not yet risky, but instead a perfect substitute of the risk-free asset.

At time 0, an unexpected innovation raises the expected fundamental value above $\overline{F}$. For simplicity, we assume that the risky asset will pay a single dividend at the maturity/payoff date t_{pay} in the amount of $d_{t_{pay}} = \overline{\overline{F}}R^{t_{pay}}$. At time 0, investors realize that the risky asset has become more valuable, but do not know the magnitude of the gains or the payoff date. Beliefs over $\overline{\overline{F}}$—which as of time 0 are the same for all investors—are captured by the cdf $\Psi(\cdot)$, with $\Psi(\overline{F}) = 0$ and $\Psi(\infty) = 1$. We also assume that the payoff date t_{pay} is an increasing function of $\overline{\overline{F}}$, i.e., that the greater the increase in fundamental value, the longer it will take for the asset to pay dividends. Hence, for any $n = 0, 1, 2, \ldots$

$$t_{pay}(\overline{\overline{F}}) = \underline{t}_{pay} + n \quad \text{if } \overline{\overline{F}} \in \left(\overline{\overline{F}}_n, \overline{\overline{F}}_{n+1}\right], \tag{13}$$

where the lower bound $\underline{t}_{pay}$ is a positive integer, $\overline{\overline{F}}_0 = \overline{F}$ and $\overline{\overline{F}}_{n+1} > \overline{\overline{F}}_n$. Clearly, at time t_{pay}, the value of $\overline{\overline{F}}$ will become known. Our focus, however, will be in situations where $\overline{\overline{F}}$ becomes known before the dividend is paid.

We assume that investors cannot borrow enough at time 0 to bid p_t up to its new expected value—we will later derive parameter conditions ensuring this. Under this assumption, the boom is not a one-time jump in price, but is instead sustained for a number of periods in which prices and credit reinforce each other. Borrowing to buy the risky asset raises its price, which in turn loosens borrowing constraints, allowing investors to borrow more and bid prices even higher. Prices catch up to the new fundamental value, and begin surpassing it at t_0. Any price gains after t_0 are not granted by fundamentals and bound to disappear at the crash time t_c. We assume that the lower bound on the payoff time $\underline{t}_{pay}$ is always larger than t_c, so that payment of dividends occurs well after the bubble and crash have played out.

A crucial assumption that we borrow from AB is that agents do not observe t_0 perfectly. Instead, every period from t_0 to $t_0 + N - 1$, immediately after observing the price, a mass $1/N$ of agents receive a signal revealing that the risky asset is no longer undervalued. Signals define N types of agents $n = t_0, ..., t_0 + N - 1$. Agents observe n, but not t_0. In other words, each agent knows she observes her signal, but not when others do. Once an agent observes her signal at time n she knows that the start of the overvaluation t_0 may have been as early as $n - (N - 1)$ or as late as n. Hence, the distribution of t_0 conditional on $n \geq N$ is given by[9]

$$\varphi(t_0|n) = \begin{cases} \frac{\Pr[t_0]}{\Pr[n-(N-1)]+\cdots+\Pr[n]} & \text{if } n-(N-1) \leq t_0 \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Since t_0 is the first period with $p_t \geq \overline{\overline{F}}$, the probability of each value of $t_0 \in \{1, 2, ...\}$ is derived from the distribution of $\overline{\overline{F}}$ as follows:

$$\Pr[t = t_0] = \Psi(p_t) - \Psi(p_{t-1}). \tag{15}$$

[9]In the special case with $t_0 < N$, types with $n < N$ know that t_0 must be above 0 since $n - (N - 1) \leq 0$, and thus condition on $t_0 \in \{1, ..., n\}$.

Sequential arrival of signals places investors along a line, but they are uncertain about their relative position in it. This plays an important role, because all investors—including those late in the line—assign positive probability to the event that they could be early.[10] If the price grows fast enough, it may be optimal to continue buying the overvalued asset, trading off the chance to resell at a profit in the event of being an early-signal investor against the risk of being a late-signal investor and getting caught in the crash.

As long as all types continue to borrow and invest as much as they can, only the θ shares forced by shocks are sold. However, as soon as shock-induced sellers are joined by sellers who anticipate a crash, the price reveals that the bubble is starting to burst, and the boom comes to an end. To fix ideas, suppose that type-n investors plan to wait for $\tau^* \geq 0$ periods and then sell at $n + \tau^*$. (We will later show that this strategy profile is an equilibrium.) While $t < t_0 + \tau^*$, $S_t = \theta$, and the boom proceeds at maximum speed as all agents borrow and invest as much as they can. This continues until, in period $t_c = t_0 + \tau^*$, the number of shares for sale increases to $S_t = \theta + (1-\theta)/N$, as shock-induced sellers are joined by sellers of type $n = t_0$. At the same time, the number of buyers falls from $1 - \theta$ to $(1-\theta)(1-1/N)$. Seeing the corresponding decline in the price ratio p_t/p_{t-1}, buyers realize that $t = t_c$, and thus also learn the value of $t_0 = t_c - \tau^*$. The boom thus concludes, with agents of type $n = t_0$ selling at the peak and all other types finding themselves in the position of the *greater fool*.

Since the crash reveals the value of t_0, in the post-crash phase agents are willing to bid only up to the fundamental value. At time $t_c + 1$, agents of type $n = t_0$ repurchase the risky asset from the losing types $n > t_0$, many of whom are in default. From that point on, agents who are in good credit and who are not hit by the shock buy the asset from those who are hit by the shock. This continues until the risky asset pays off and disappears at time t_{pay}. The full timeline is shown in Figure 1.

[10]This specification of signals is simply a discretized version of that in Abreu and Brunnermeier (2003). In experimenal work, Moinas and Pouget (2012) also introduce uncertainty about the order in which agents move, so that the last buyer thinks that she may not be the last. Other models with a 'greater fool' flavor include Allen and Gorton (1993), Allen, Morris and Postlewaite (1993) and Conlon (2004).

3 Equilibrium Concept and Strategies

3.1 Equilibrium

The equilibrium concept is Perfect Bayesian Equilibrium (PBE) which consists of mutually consistent strategies and beliefs. Investor i's strategy of agent i is a plan of actions $a_{i,t} = \{m_{i,t}, s_{i,t}, \xi_{i,t}\}$ for all t, contingent on information $I_{i,t}$. In equilibrium, for all agents, at all times, and under every possible contingency, beliefs must be consistent with strategies and actions must be optimal given beliefs. Information $I_{i,t}$ includes $p^{t-1} = \{\ldots, p_{t-2}, p_{t-1}\}$, the price history up to period $t-1$, as well as the discount factor $\delta_{i,t}$, since the agent knows the realization of the preference shock. Once the agent has observed the overvaluation signal, i.e., once $t \geq \upsilon(i)$, the signal $\upsilon(i) = n$ is added to $I_{i,t}$. In sum, $I_{i,t}$ is given by $\{p^{t-1}, \delta_{i,t}\}$ if $t < \upsilon(i)$ and by $\{p^{t-1}, \delta_{i,t}, \upsilon(i)\}$ if $t \geq \upsilon(i)$.

Equilibrium beliefs $\mu_{i,t}$ are probability distributions over values of t_0, which are updated according to Bayes' rule as signals and prices are observed. Given $I_{i,t}$, the set of possible values of t_0 is given by $\mathit{supp}_{i,t}(t_0)$. Before signals are observed, $\mathit{supp}_{i,t}(t_0) = \{1, 2, 3, \ldots\}$. Observing the signal $\upsilon(i)$ allows investor i to exclude values of t_0 below $\upsilon(i) - (N-1)$ and above $\upsilon(i)$, narrowing the support down to $\mathit{supp}_{i,t}(t_0) = \{\max\{\upsilon(i) - (N-1), 1\}, \ldots, \upsilon(i)\}$. Moreover, if trading strategies depend on signals, agents to continue to learn and eliminate values from $\mathit{supp}_{i,t}(t_0)$ as they observe prices. Among the values in $\mathit{supp}_{i,t}(t_0)$, probabilities are distributed according to Bayes' rule. Hence, the equilibrium beliefs are given by

$$\mu_{i,t}(t_0) = \frac{\Pr[t_0]}{\sum\limits_{\tau_0 \in supp_{i,t}(t_0)} \Pr[\tau_0]}. \tag{16}$$

Given $(h_{i,t}, b_{i,t}, l_{i,t}, \varsigma_{i,t})$ and $I_{i,t}$, investor i's choices $(m_{i,t}, s_{i,t}, \xi_{i,t})$ solve

$$V_{i,t}(h_{i,t}, b_{i,t}, l_{i,t}, \varsigma_{i,t}) = \max_{(m_{i,t}, s_{i,t}, \xi_{i,t})} E\left[V_{i,t+1}(h_{i,t+1}, b_{i,t+1}, l_{i,t+1}, \varsigma_{i,t+1}) | I_{i,t}\right] \tag{17}$$

subject to (2), (3), $(m_{i,t}, s_{i,t}) = (0, h_{i,t})$ if $\varsigma_{i,t} = 0$, (5), (6), $\xi_{i,t} \in [0, 1]$, and

$$
\begin{pmatrix} \varsigma_{i,t+1} \\ l_{i,t+1} \\ c_{i,t} \\ b_{i,t+1} \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ R\left(l_{i,t} - \widetilde{b}_{i,t}\right) \\ 0 \\ 0 \end{pmatrix} & \text{if } Rl_{i,t} > R\widetilde{b}_{i,t} + \phi p_t h_{i,t+1} \\ \begin{pmatrix} 1 \\ \phi p_t h_{i,t+1} \\ \xi_{i,t}\left(\widetilde{b}_{i,t} + l_{i,t+1} - Rl_{i,t}\right) \\ \left(1 - \xi_{i,t}\right)\left(\widetilde{b}_{i,t} + l_{i,t+1} - Rl_{i,t}\right) \end{pmatrix} & \text{if } Rl_{i,t} \leq R\widetilde{b}_{i,t} + \phi p_t h_{i,t+1}. \end{cases}
$$

Finally, equilibrium also requires market clearing

$$
H_t = 1. \tag{18}
$$

3.2 Strategies

Investors in default (i.e., with $\varsigma_{i,t} = 0$) have very little freedom. They must liquidate assets and forego endomwents until repaying their debt. Specifically, their strategy is given by

$$
(m_{i,t}, s_{i,t}, \xi_{i,t}) = \begin{array}{ll} (0, h_{i,t}, 0) & \text{if } \delta_{i,t} = 1/R \\ (0, h_{i,t}, 1) & \text{if } \delta_{i,t} = 0. \end{array} \tag{19}
$$

The choice of $\xi_{i,t} = 1$ if $\delta_{i,t} = 0$ means that agent i wishes to consume 100% of the balance available for consumption. Said balance, however, is only positive in the period when she fully repays and regains good credit, i.e., when $l_{i,t+1} = 0$ and $\varsigma_{i,t+1} = 1$. As long as $l_{i,t+1} > 0$ and $\varsigma_{i,t+1} = 0$, the balance available for consumption is zero, and thus $c_{i,t} = 0$ regardless of the choice of $\xi_{i,t}$.

Investors in good credit can freely choose to buy or sell the risky asset, a choice which is nontrivial when they are not hit by the preference shock. Before and after the boom, investors simply bid the asset to its fundamental value. To do this, they bid a fraction $\theta f_t / [(1-\theta)(B_t + y_t)]$ of their available risk-free funds $b_{it} + y_t$. The strategic interaction of interest occurs during the boom. Following Abreu and Brunnermeier (2003) and Doblas-Madrid (2012), we focus on trigger strategies, in which type-$\upsilon(i)$ investors plan to ride the bubble for τ^* periods after observing their signal and sell at time $\upsilon(i)+\tau^*$. If all agents follow this strategy, the bubble is pricked when agents of the first type $\upsilon(i) = t_0$ sell at time $t_c = t_0 + \tau^*$. In sum, strategies of agents in good credit are given by:

1. If hit by the preference shock, $\delta_{i,t} = 0$:

$$(m_{i,t}, s_{i,t}, \xi_{i,t}) = (0, h_{i,t}, 1).$$

2. If not hit by the preference shock, $\delta_{i,t} = 1/R$:

$$(m_{i,t}, s_{i,t}, \xi_{i,t}) = \begin{cases} \left(\frac{\theta \overline{F} R^t}{(1-\theta)(B_t+y_t)} \left[b_{it} + y_t\right], 0, 0\right) & \text{if } t < 0 \\ (b_{i,t} + y_t, 0, 0) & \text{if } 0 \leq t < \min\{\upsilon(i) + \tau^*, t_c + 1\} \\ (0, h_{i,t}, 0) & \text{if } t = \upsilon(i) + \tau^* < t_c + 1 \\ \left(\min\left\{1, \frac{\theta \overline{\overline{F}} R^t}{(1-\theta)(B_t+y_t)}\right\} \left[b_{it} + y_t\right], 0, 0\right) & \text{if } t_c + 1 \leq t < t_{pay} \\ (0, 0, 0) & \text{if } t \geq t_{pay} \end{cases}$$

To derive equilibrium beliefs $\mu_{i,t}$ from trigger strategies, note that agent i learns about t_0 from the signal $\upsilon(i)$ and the crash $t_c = t_0+\tau^*$. From the signal, she infers that t_0 is between $\upsilon(i)-(N-1)$ and $\upsilon(i)$. Given this, strategies imply that t_c must be between $\upsilon(i) - (N-1) + \tau^*$ and $\upsilon(i) + \tau^*$. (Assuming that $\tau^* \geq N-1$, so that agent i has observed the signal as of $\upsilon(i)-(N-1)+\tau^*$.) When she observes the price at time $\upsilon(i) - (N-1) + \tau^*$, agent i learns that $t_0 = \upsilon(i) - (N-1)$ if the bubble bursts, or that $t_0 > \upsilon(i) - (N-1)$ if it does not. Prior to observing this price, she assigns a given probability to the event that $t_0 = \upsilon(i) - (N-1)$. Every period after that, she either gets caught in the crash and learns the value of t_0 or, if the bubble continues, she discards another value of t_0 from $\mathit{supp}_{i,t}(t_0)$. As she discards successive values, the crash probability rises. In fact, if t_0 happens to be $\upsilon(i)$, agent i learns the true value of t_0 when she observes the price at time $\upsilon(i) - 1 + \tau^*$, which allows her to eliminate the possibility that $\upsilon(i) - 1$.

4 Equilibrium in the benchmark case

Under some conditions, our model endogenously generates the same dynamics as in Doblas-Madrid (2012). The specific conditions under which this is the case are the following:

1. Parameters and the realization of t_0 are such that the price growth rate during the bubbly phase of the boom closely approximates a constant rate $G > R$.

2. Once t_0 is known, the expectation $E[\overline{\overline{F}}|t_0]$ equals Gp_{t_0}.

3. As the price growth rate approximates G, the distribution of t_0 approximates a geometric distribution. In other words, the cdf Ψ governing the distribution of $\overline{\overline{F}}$ satisfies the property that, if $p_{t+1} = Gp_t$,

$$\frac{\Pr[t+1 = t_0]}{\Pr[t = t_0]} = \frac{\Psi(p_t) - \Psi(p_{t-1})}{\Psi(p_{t-1}) - \Psi(p_{t-2})} = \lambda, \tag{20}$$

   for some $\lambda \in (0, 1)$.

4. The price does not overshoot below fundamental value. In other words, buyers after the crash have or can borrow enough of the risk-free asset to buy all the shares offered for sale—which might spike due to forced liquidations—at a price equal to fundamental value.

The case in which conditions 1-4 are satisfied serves as a useful benchmark for two reasons. The first is that conditions 1-4 eliminate a series of complications, greatly simplifying the analytical derivation of equilibrium bubble duration. Second, in this benchmark case, we can apply some of the results from Doblas-Madrid (2012), subject to only minor modifications. As we will in a later Section, if conditions 1, 2, or 3 fail, formulas become more cumbersome, but the gist of the analysis remains similar. Considering what happens when condition 4 fails is more involved, although the effort will pay off in terms of additional results.

4.1 Before the fundamental shock

The pre-boom phase before period 0 is a quiet one, in which the price p_t simply grows at the risk-free rate. Agents hit by preference shocks sell their entire holdings of the risky asset. Since shocks are i.i.d., the mass of shares offered for sale always amounts to $S_t = \theta$. Those who are not hit by the shock are willing to buy these shares at a price equal to fundamental value $\overline{F}R^t$. Of course, for this to be possible in equilibrium, buyers must wield enough—owned or borrowed—funds to bid $M_t = \theta\overline{F}R^t$. We therefore assume that, at all times

$$\theta\overline{F}R^t \leq (1-\theta)\left(B_t + y_t\right), \tag{21}$$

where $(1-\theta)$ is the measure of buyers and B_t+y_t the aggregate risk-free balances available when the asset market opens. Also note that B_t includes funds borrowed in the previous period in the amount of $L_t = \phi\overline{F}R^{t-1}$.

With the risky asset appreciating at the rate R, all borrowers can repay or roll over their debts at the debt refinancing stage. Sellers repay θRL_t and consume, in the aggregate

$$C_t = \theta\left[R\left(B_t + y_t\right) + \overline{F}R^t(1-\phi)\right]. \tag{22}$$

Buyers pledge the risky asset—of which they now own the full supply—to borrow $L_{t+1} = \phi\overline{F}R^t$ and pay back their share of last period's debt $(1-\theta)RL_t = (1-\theta)\phi\overline{F}R^t$. Since buyers do not consume, aggregate risk-free balances evolve according to

$$\begin{aligned} B_{t+1} &= (1-\theta)R(B_t + y_t - M_t) - (1-\theta)RL_t + L_{t+1} \\ &= (1-\theta)(B_t + y_t) - \theta(1-\phi)\overline{F}R^t. \end{aligned} \tag{23}$$

We assume that, as of time 0, investors have more wealth than they need to purchase the risky asset, i.e., we assume that as of time 0, (21) holds with strict inequality.

To fix ideas, consider an example where endowments grow at the risk-free rate, i.e., $y_t = \overline{y}R^t$. Then, (23) becomes

$$B_{t+1} = (1-\theta)RB_t + \left[(1-\theta)\overline{y} - \theta(1-\phi)\overline{F}\right]R^t. \tag{24}$$

Iterating, and assuming that $(1-\theta)R < 1$, risky balances also grow at the risk-free rate and are given by

$$B_t = \left[\overline{y} - \overline{F}(1-\phi)\theta/(1-\theta)\right]R^t \tag{25}$$

and (21) holds with strict inequality as long as $\overline{y} > \overline{F}[\theta/(1-\theta) - \phi]$.

4.2 Innovation, boom and bubble

At the beginning of period 0, before the asset market opens, an unanticipated fundamental shock raises the value of the risky asset's future dividends. The value of such dividends—discounted to time 0—increases from $\overline{F}$ to $\overline{\overline{F}}$. As of time 0, agents know that the fundamental value of the risky asset has increased substantially, but they do not know exactly by how much, i.e., they do not know the precise magnitude of $\overline{\overline{F}}$. Their beliefs over possible values of $\overline{\overline{F}}$ are captured by the cdf Ψ. We assume that, for a number of periods, the expected $\overline{\overline{F}}$ exceeds the maximum price that buyers are able to pay given their savings, endowments and ability to borrow. At time 0, the mass θ of agents hit by preference shocks submit orders to sell $S_0 = \theta$ shares, while the remaining mass $(1-\theta)$ bid as much as they can for the risky asset. Hence, $M_0 = (1-\theta)[B_0 + y_0]$ and the price p_0 is given by

$$p_0 = \tfrac{(1-\theta)}{\theta}[B_0 + y_0].$$

The fact that p_0 must be greater than $\overline{F}$ follows from our assumption that (21) holds with strict inequality at time 0.

At the debt refinancing stage, sellers repay a fraction θ of RL_0 and consume. Buyers repay the remaining fraction $1-\theta$ of the outstanding debt, and—since they own the entire supply of the risky asset—borrow the full new amount $L_1 = \phi p_0$ and thus begin period 1 holding $B_1 = L_1 - (1-\theta)RL_0$ units of the risk-free asset. The same pattern of actions taken in period 0 is repeated in period 1, with the only sellers being those hit by the preference shock and all other agents investing as much as they can into the risky asset. Therefore,

$$p_1 = \tfrac{(1-\theta)}{\theta}\left[(L_1 - (1-\theta)RL_0) + y_1\right].$$

and $L_2 = \phi p_1$.Iterating, this process gives rise to a recursion defined by

$$p_{t+1} = \frac{1-\theta}{\theta}\left[(L_{t+1} - (1-\theta)\,RL_t) + y_{t+1}\right], \tag{26}$$

and $L_{t+1} = \phi p_t$. Substituting the latter into (26) and dividing through by p_t, we obtain

$$\frac{p_{t+1}}{p_t} = \frac{1-\theta}{\theta}\left[\phi - (1-\theta)R\phi\frac{p_{t-1}}{p_t} + \frac{y_{t+1}}{p_t}\right].$$

We next define $G_t \equiv p_t/p_{t-1}$, and conjecture that this process generates price growth rates higher than R. Under this conjecture, and the assumption that endowments do not grow faster than R, the term y_{t+1}/p_t approaches 0 as t grows, and we can rewrite the above as

$$G_{t+1} = \frac{1-\theta}{\theta}\phi \left[1 - \frac{(1-\theta)R}{G_t}\right]. \tag{27}$$

If this process converges to a constant growth rate $G_{t+1} = G_t = G$, such rate must solve the quadratic equation

$$G^2 - \tfrac{1-\theta}{\theta}\phi G + \tfrac{(1-\theta)^2}{\theta}\phi R = 0,$$

which has only one stable root given by

$$G = \tfrac{1-\theta}{2\theta}\phi \left[1 + \sqrt{1 - 4\theta R/\phi}\right].$$

If $4\theta R \leq \phi$, a constant rate G exists. After a number of periods, the price growth rate converges to this constant level, which increasing in ϕ and decreasing in θ and R. It is not difficult to show that whenever G exists, it must be greater than R.[11]

In the event that $4\theta R > \phi$, the constant rate G does not exist. Simulation results depicted in Figure 2 show rapidly growing prices and debt for a number of periods, followed by declining and eventually negative growth rates. As we will discuss in a later Section, one can still generate bubbles in this case, as long as the bursting date arrives before price growth slows down, although it is necessary to consider strategies different from trigger strategies. For the remainder of this section, however, we will focus on the case where $4\theta R \leq \phi$, so that a constant rate G does exist.

Recapitulating, during boom periods $t \in \{0, \ldots, t_c - 1\}$, all agents borrow and invest as much as they can into the risky-asset, with the only sales being those forced by preference shocks, $S_t = \theta$. Under the assumption that $4\theta R \leq \phi$, the growth rate of the price generated by the borrowing and reinvesting recursion defined above converges towards a constant rate G.

The boom ends at time t_c, when shock-induced sellers are joined by sellers who are leaving the market in anticipation of the crash, and thus S_{t_c} becomes $\theta + (1-\theta)/N$. At the same time,

[11]To see this, note that if $\phi = 4\theta R$, $G = 2(1-\theta)R$. In this case, $G > R$ is equivalent to $\theta < 1/2$, which must be the case since $4\theta R = \phi \leq 1$. If $\phi > 4\theta R$, G the difference between G and R grows.

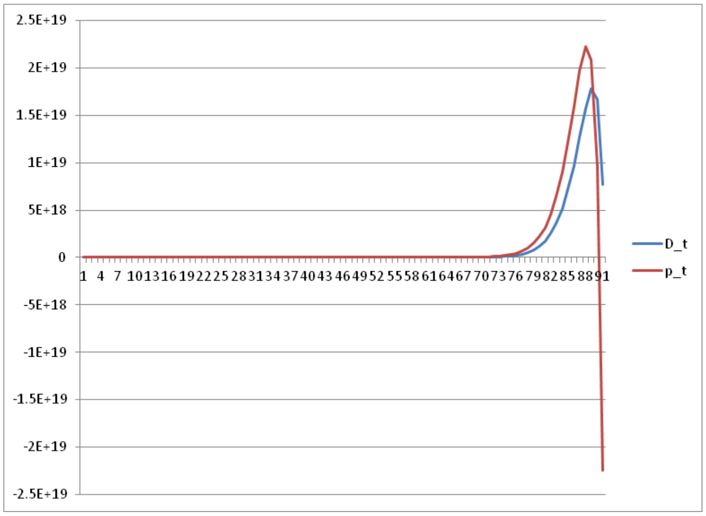

Figure 2: Prices and debt when G does not exist.

the number of buyers falls from $1-\theta$ to $(1-\theta)(1-1/N)$. These developments lead to a decline in the price growth rate, which pricks the bubble and triggers the crash. Since agents know that $t_c = t_0 + \tau^*$, the value of t_0 becomes common knowldege when the price p_{t_c} is observed. It is important to note that, since orders to buy and sell are submitted in Step 1 of the asset market, buyers do not learn how many sellers there are until the market clears in Step 2. By that point, however, it is too late for them to cancel their buy orders.

To compute the decline in the price growth rate at time t_c, we begin by writing the price as

$$p_{t_c} = \frac{(1-\theta)(1-\frac{1}{N})}{\theta + \frac{1-\theta}{N}} \left(\phi p_{t_c-1} - (1-\theta)\, R\phi p_{t_c-2} + y_{t_c}\right)$$

and dividing by p_{t_c-1} to obtain

$$\frac{p_{t_c}}{p_{t_c-1}} = \frac{\frac{(1-\theta)(1-\frac{1}{N})}{\theta+\frac{1-\theta}{N}}}{(1-\theta)/\theta} \left[\frac{\phi p_{t_c-1} - (1-\theta)\, R\phi p_{t_c-2}}{\phi p_{t_c-2} - (1-\theta)\, R\phi p_{t_c-3} + y_{t_{c-1}}} + \frac{y_{t_c}}{p_{t_c-1}}\right].$$

If the boom is long enough, we can ignore y_{t_c}/p_{t_c-1} and rearrange terms as follows

$$\frac{p_{t_c}}{p_{t_c-1}} = \frac{\theta(N-1)}{1+\theta(N-1)} \left[\frac{\phi\frac{p_{t_c-1}}{p_{t_c-3}} - (1-\theta)\, R\phi\frac{p_{t_c-2}}{p_{t_c-3}}}{\phi\frac{p_{t_c-2}}{p_{t_c-3}} - (1-\theta)\, R\phi + \frac{y_{t_{c-1}}}{p_{t_c-3}}}\right].$$

Similarly, we can approximate $y_{t_c}/p_{t_c-3} \simeq 0$ and $p_t \simeq Gp_{t-1}$, allowing us to simplify the above expression to obtain

$$p_{t_c} = p_{t_{c-1}} \frac{\theta(N-1)}{1+\theta(N-1)} G. \tag{28}$$

The decline in the price growth rate, in addition to revealing information, reduces sellers' revenue as captured by the fraction $\theta(N-1)/[1+\theta(N-1)]$. This revenue effect is decreasing in θ and N, and can be made arbitrarily small by increasing N.[12] Nevertheless, as we will see shortly, it is not necessary for this effect to be negligible in order to generate bubbles. What is necessary is that, overall, the effect does not reduce the potential reward from riding the bubble, in terms of capital gains, to such a degree that agents become unwilling to bear the associated crash risk.

Before moving on to the collapsing asset market of period $t_c + 1$, it is useful to track how agents' portfolios evolve over the course of period t_c. Agents of all types start the period holding on average assets $H_{t_c} = 1$ and $B_{t_c} = \phi p_{t_{c-1}} - (1-\theta)R\phi p_{t_c-2}$, and a liability $L_{t_c} = \phi p_{t_{c-1}}$. Those hit by preference shocks collectively sell θ shares, repay their share debt $\theta R\phi p_{t_{c-1}}$ and consume $C_{t_c} = \theta R[p_{t_c} + y_{t_c} - (1-\theta)R\phi p_{t_c-2}]$. Since they do not hold any shares of the risky asset, they do not borrow. The mass $(1-\theta)/N$ of type-t_0 agents who sell without being hit by the shock also finish period t_c without any risky asset shares and without any debt. However, instead of consuming, they save their holdings of the risk-free asset, which amount on average to $R[p_{t_c} + y_{t_c} - (1-\theta)R\phi p_{t_c-2}]$. Finally, the hapless mass $(1-\theta)(1-1/N)$ of agents who are buyers at time t_c find themselves at the end of the period holding the entire supply of the risky asset. Since the price ratio $p_{t_c}/p_{t_{c-1}}$ is still above R, they can still borrow ϕp_{t_c}, repay their debts $(1-\theta)(1-1/N)R\phi p_{t_{c-1}}$, and avoid default for the moment.

4.3 Crash and aftermath without overshooting

Since all agents know that $t_c = t_0 + \tau^*$, the price p_{t_c} reveals the true value of t_0. This puts an end to the asymmetry of information that began when signals arrived. Once t_0 becomes common knowledge, all agents compute the same expected fundamental value $E[\overline{\overline{F}}|t_0]$, which given the definition of t_0, must be between p_{t_0} and p_{t_0-1}. Condition 2 allows us to further simplify the analysis by equating $E[\overline{\overline{F}}|t_0]$ with p_{t_0}/G.

In the benchmark case, which most closely resembles Doblas-Madrid (2012), price falls to

[12] Such a large-N assumption is in place for most of the analysis in Doblas-Madrid (2012).

fundamental value in the crash, but not further. That is, under conditions 4 and 2,

$$p_{t_c+1} = E[\overline{\overline{\overline{F}}}|t_0]R^{t_c+1} = \frac{p_{t_0}}{G}R^{t_c+1}.$$

For this price to arise from trading in the asset market, it must also be the case that

$$M_{t_c+1} = p_{t_c+1}S_{t_c+1},$$

which is only possible if buyers at time $t_c + 1$ have enough funds to acquire all of the S_{t_c+1} shares sold. This is possible in two scenarios. In the first, the degree of leverage ϕ is sufficiently small that agents who bought the risky asset at t_c easily avoid defaulting at $t_c + 1$ by selling a few of their shares. In other words, the first scenario is one in which—even after crash losses—debt is still a relatively small fraction of the portfolio's value. In that case, there are no defaults and no glut of forced liquidations of repossessed risky shares. Agents who bought the risky asset at t_c reduce their debt to bring it back within the allowed limit by selling a number of shares, which depends on ϕ. On the other hand, the resources available for the aggregate M_{t_c+1} depend on the sales revenue obtained by the fortunate type-t_0 agents who correctly timed the market at t_c. If the number of sales for sale does not increase, type-t_0 agents—specifically the fraction of them not hit by the preference shock—are able and willing to pay a price p_{t_c+1} since they sold at a the higher price p_{t_c}. In scenario 1, there is no overshooting because ϕ is small enough that the increase in shares for sale S_{t_c+1} does not outweigh the drop in price.

In the second scenario, the bubble is so large that the speculative profits of type-t_0 agents outweigh any increase in the number of shares for sale. The maximum possible increase in shares for sale occurs when every buyer at t_c must liquidate all her shares at time $t_c + 1$. In that event, the number of shares for sale increases from $S_{t_c} = \theta + (1 - \theta)/N$ to $S_{t_c+1} = 1$. Nevertheless—as we will see shortly—for appropriate parameters bubble duration τ^* can be made arbitrarily large, so that the price falls by a greater percentage than the number of sales for sale increase.

In each of these two scenarios, the price of the asset falls to fundamental value at the time of the crash and then grows at the risk-free rate until the dividend is paid at the payoff date t_{pay}.

4.4 Characterizing bubble duration

Under conditions (1)-(4), the characterization of equilibrium τ^* is derived from the conditions verifying that agents—unless forced to sell by preference shocks—are willing to invest as much as they can into the risky asset and sell all their shares at time $\upsilon(i) + \tau^*$. Since agents hit by preference shocks have zero discount factors, it is straightforward that they are willing to sell and consume. Thus, our focus shall be on the actions of agents who are not hit by preference shocks during boom periods. It is the choices made under these circumstances that determine whether bubbles can arise and how large can they become. Equilibrium requires that agents be willing to: *(i)* Sell when the strategy dictates that they should sell and *(ii)* Wait when the strategy dictates that they should wait. Part *(i)* holds for any τ^*. To see why, note that if agent knows that others of her same type are selling at t, she knows that the price will show reveal the sales, and the bubble will burst next period.

Part *(ii)* is less obvious and more important, since it pins down how large bubbles can become in equilibrium. To understand the choices of waiting agents, consider a type-$\upsilon(i)$ agent. She learns about t_0 from signal and prices. From the signal, she infers that the support of t_0 is the set $\{\upsilon(i) - (N-1), \ldots, \upsilon(i)\}$. Under condition 3, the conditional probabilities are given by

$$\Pr[t_0 = \tau | \upsilon(i)] = \frac{\lambda^{\tau - [\upsilon(i) - (N-1)]}}{1 + \lambda + \lambda^2 + \cdots + \lambda^{N-1}}$$

An agent of type $\upsilon(i)$ eliminates values from her support of t_0 as she observes that the bubble does not burst at times $\upsilon(i) - N + 1 + \tau^*$, $\upsilon(i) - N + 2 + \tau^*$, and so forth. If she happens to be "first in line", i.e., if $\upsilon(i) = t_0$ she finally learns that $t_0 = \upsilon(i)$ when she observes the price at time $\upsilon(i) - 1 + \tau^*$. In equilibrium, of course, she must be willing to wait until the scheduled selling date $\upsilon(i) + \tau^*$, although the probability of a crash conditional on the bubble not having burst increases every period as this date nears. Hence, preemptively sales are most tempting at time $\upsilon(i) - 1 + \tau^*$, just one period before the selling time dictated by the strategies $\upsilon(i) + \tau^*$. If agent i is willing to wait at time $\upsilon(i) - 1 + \tau^*$, she will also be willing to wait at all other times.

At time $\upsilon(i) - 1 + \tau^*$ this point, agent i may sell preemptively or wait one more period. If she sells, she will avoid the crash and sell at a price which varies slightly depending on t_0. If $t_0 = \upsilon(i) - 1$ shock-induced agents and type-$\upsilon(i) - 1$ agents will sell and the price will be reduced by a factor $\theta(N-1)/[1+\theta(N-1)]$ and if $\upsilon(i)-1$ only shock-induced sellers will sell and the price will be accordingly higher. If agent i decides to wait, she will get caught in the crash in the event that $t_0 = \upsilon(i)-1$, and otherwise ride the bubble for one more period, selling at the price $\theta(N-1)/[1+\theta(N-1)]p_{t_0}G^{\tau^*+1}$. In sum, waiting is optimal if

$$\left(\frac{\theta(N-1)}{1+\theta(N-1)}+\lambda\right)p_{t_0}RG^{\tau^*} \leq \frac{p_{t_0}}{G}R^{\tau^*+1} + \lambda\frac{\theta(N-1)}{1+\theta(N-1)}p_{t_0}G^{\tau^*+1}$$

Dividing through by $p_{t_0}RG^{\tau^*}$ and rearranging terms, we can rewrite the above 'sell-or-wait' inequality as

$$\frac{\theta(N-1)}{1+\theta(N-1)}+\lambda \leq \left(\frac{G}{R}\right)^{-(\tau^*+1)} + \lambda\frac{\theta(N-1)}{1+\theta(N-1)}\frac{G}{R} \tag{29}$$

To read this inequality, it is easiest to first note that, when N is large enough that $\theta(N-1)/[1+\theta(N-1)]$ is close to one, it simplifies to

$$1 \leq \frac{1}{1+\lambda}\left(\frac{G}{R}\right)^{-(\tau^*+1)} + \frac{\lambda}{1+\lambda}\frac{G}{R}.$$

In this reduced version, the inequality governing the sell-vs-wait choice says that, relative to selling, waiting yields crash losses with probability $1/(1+\lambda)$ and one more period of appreciation with probability $\lambda/(1+\lambda)$. Thus, the higher λ and G/R, the greater the bubble duration τ^* that can be supported in equilibrium. The interpretation of the more general inequality (29) is in essence the same. When $\theta(N-1)/[1+\theta(N-1)]$ is not close to one, the inequality changes quantitatively as the expressions for the payoffs become more cumbersome, but the qualitative interpretation remains the same. From (29), we can characterize the set of values of τ^* that can be supported in equilibrium. We summarize our findings in Proposition 1 below, which is an adaptation of the one in Doblas-Madrid (2012).

Proposition 1 *Suppose that conditions (1)-(4) hold and that agents follow trigger strategies, with type-$\upsilon(i)$ agents planning to sell at time $\upsilon(i)+\tau^*$. Then: (a) If $G/R < [1+\theta(N-1)]/[\theta(N-1)]+1/\lambda$, equilibrium can be supported for any τ^* between 0 and $-1-\ln(\{\theta(N-1)/[1+\theta(N-1)]\}+(1-\lambda\{\theta(N-1)/[1+\theta(N-1)]\}G/R))/\ln(G/R)$. (b) If $G/R \geq [1+\theta(N-1)]/[\theta(N-1)]+1/\lambda$, any integer $\tau^* \geq 0$, can be supported in equilibrium.*

The proof follows the same steps as the proof of Proposition 1 in Doblas-Madrid (2012). The upper bound on τ^* given by $-1 - \ln(\{\theta(N-1)/[1+\theta(N-1)]\} + (1 - \lambda\{\theta(N-1)/[1+\theta(N-1)]\}G/R))/\ln(G/R)$ is derived directly from (29). The proof in Doblas-Madrid (2012) also discusses, for the sake of completeness, the complications that arise in cases where $v(i) < N$, or $\tau^* < N-1$. If $v(i) < N$, agent i's signal is more informative than signals greater or equal to N, since $v(i) - (N-1)$ and possibly other values can be eliminated from the support of t_0 given the knowledge that t_0 must be positive. In the event that $\tau^* < N-1$, the bubble bursts before all agents have observed their signals. This is not probematic, since agents who have yet to observe signals assign positive probability to an infinite set of values of t_0.

In equilibria with $\tau^* > N$, the model generates *strong bubbles* in the sense of Allen et al. (1993). That is, by period $t_0 + N$, agents of all types know that the risky asset is overvalued, and are nevertheless willing to continue buying it. Before period $t_0 + N$, some agents still believe that the boom may be fundamental, and thus, bubbles with $\tau^* \leq N$ are in a sense weaker.

5 Discussion and Extensions

In Doblas-Madrid (2012), the shock raising expected future dividends is assumed to coincide with the exogenous acceleration of endowment growth. In this paper, we replace exogenous endowment growth with an endogenous self-reinforcing feedback loop between asset prices and collateralized credit. In addition to addressing a limitation in previous work, the addition of a credit market allows us to analyze the effect of lending policies on speculative bubbles. Under conditions 1-4, the model is tractable enough to yield some analytical results. The self-reinforcing cycle of borrowing and investing endogenously yields price growth rates that converge to a constant. This constant exceeds the risk-free rate and depends positively on the maximum loan-to-value ratio allowed by the lender. Since maximum bubble duration depends on the speed of bubble growth, loosening lending standards (as measured by high loan-to-value ratios) is conducive to bubbles, and tighter credit can reduce or even eliminate bubbles. However, this relationship between lending standards and bubbles is not monotonic. If borrowing limts are loosened to include the entire present value of future endowments, instead of collateral values, bubbles do not arise, since there is no continued growth in the resources that can be invested in the bubble.

In fact, borrowing constraints expanding alongside asset prices are an essential ingredient allowing us to generate bubbles and circumvent some well-known results ruling out speculative trading. Specifically, in Tirole (1982), agents can borrow as much as they need in order to bid for the asset. Thus, the price is basically an average of all agents' valuations. Knowing that others' valuations are as likely to be right as their own, agents are unwilling to trade if they are risk averse, and indifferent between trading and not under risk neutrality. Let us contrast this with a bubbly equilibrium presented above. Consider periods after all agents have observed the signal, but before the crash. All agents would like to bid more for the asset, but cannot access the funds to do so. Thus, the price reflects the amounts agents can invest, not their estimates of the asset's worth. Even at time t_c, buyers strictly prefer to buy because—according to the information available to them—potential gains if the bubble continues outweigh the crash risk. Thus, risk neutrality—while helpful for tractability purposes—is not essential. If agents could borrow, the bubble would not arise for two reasons. First, instead of growing at the rate G for a number of periods, the price would jump and then grow at the risk-free rate. Without the reward of growth at the rate agents would not be willing to incur the crash risk. Second, the price would be an average of agents' valuations, thus revealing the value of t_0 and eliminating the uncertainty about the bursting date.

Another assumption that differs from Tirole (1982) is the trading protocol under which that agents make buy/sell decisions in a first stage and observe prices only in a second stage. This allows type-t_0 agents to sell their shares in the period of the crash, since other agents commit to buying in the first stage of the period and cannot withdraw their bids after observing the price. This Shapley-Shubik trading protocol resembles the protocol in an exchange with market orders, which are always filled with certainty, but without a price guarantee. Under Walrasian timing—or in a market with limit orders—our results would not obtain, since buyers at t_c would be able to condition their purchase on the price. Only if there was enough noise in the price process, this assumption could be relaxed, since buyers would not be able to distinguish the beginning of the crash from random day-to-day fluctuations. Doblas-Madrid (2012) incorporates some of these ideas, although he does not allow enough noise to allow for Walrasian timing.

Two deviations from the benchmark case are worth discussing. First is the case in which the price overshoots as it crashes, an observation that is reminiscent of recent events in US housing and equity markets. This occurs for bubbles of an intermediate duration, between the two scenarios described in the previous section. That is, overshooting occurs if bubbles are large enough to force a glut of defaults and liquidations but not large enough to generate such profits among early sellers that they can acquire all the shares offered for sale at fundamental value. In the presence of overshooting, after the initial glut of sales, the number of shares for sale falls back to θ and then continues to grow at the rate R. Characterizing equilibrium bubble duration τ^* in the presence of overshooting requires adapting inequality (29) to account for the greater losses in the event that the market crashes, and the opportunity of early sellers to profit twice by reentering the market at below-fundamental prices.

Finally, it remains to discuss the situation in which the boom does not converge to a constant growth rate G. Bubbles can still arise in this case, although the simple trigger strategies consisting of waiting for τ^* periods before selling are not a Perfect Bayesian Equilibrium. To see why, note that in equilibrium agents must believe that those who observe the signal after themselves are also willing to wait for τ^* periods, and it must be possible to iterate this argument forward indefinitely. However, if there is a common-knowledge time at which price growth will slow down, the iteration eventually collides against this. This problem can be circumvented by assuming that there is a maximum possible value of $\overline{\overline{F}}$ and hence of t_0, and by specifying strategies such that τ^* decreases for agents who observe signals later. In fact, for agents observing signals at the maximum possible value of t_0, it must be that $\tau^* = 0$. This same argument would apply to the situation in which the risk-free rate was sensitive to the amount borrowed, which would be a sensible assumption for long booms. The increase in the risk-free rate would lead to a common-knowledge slowdown date, ruling out equilibria with simple trigger strategies.

6 Conclusion

We extend the rational model of speculative bubbles presented by Doblas-Madrid (2012) by adding a credit market as a source of funds that fuels bubbles. Due to borrowing constraints, initial asset prices limit how much agents can borrow, which in turn limits the immediate response of asset prices to a shock. Adjustment to the shock—instead of a one-time jump—takes place over a number of periods in which booming prices and credit reinforce each other. Under asymmetric information a la Abreu and Brunnermeier (2003), rational agents continue bidding prices past fundamental values, inflating a bubble. The ensuing crash redistributes wealth from unlucky investors, saddled with debt even after liquidating their assets to repay debts, to lucky ones, who benefit twice by selling at the top, and in some instances from re-entering the market at fire-sale below-fundamental prices. By reducing the speed at which prices grow, monetary and credit policies can shorten, or even eliminate bubbles. Specifically, such policies include raising interest rates, lowering loan-to-value ratios and loan-to-income ratios. While these policies seem intuitively desirable, our model falls short of having implications for otpimal policy. Risk neutrality and fixed supply of the risky asset pay huge dividends in terms of tractability, but they also imply that, in our model, there is no misallocation of productive resources and that the zero-sum redistribution caused by bubbles is welfare neutral. Extending the model to overcome these limitations is, in our view, a worthwhile objective for future work.

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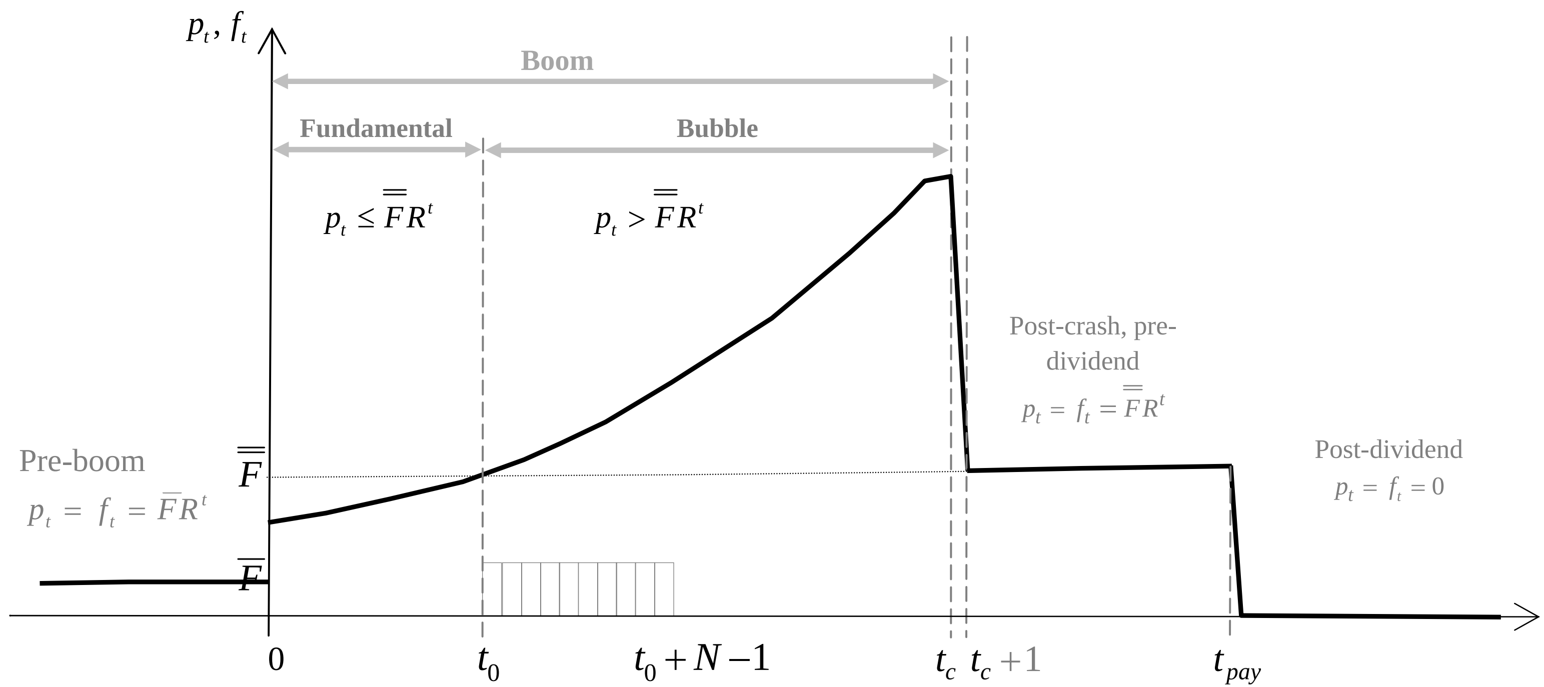

Figure 1 — The boom starting at time 0 is at first fundamental, but turns into a bubble at the imperfectly observed time t_0. Signals arrive at $t = t_0, \ldots, t_0 + N - 1$. Bubble duration τ^* will be endogenously determined in equilibrium, and the bubble will burst at $t_c = t_0 + \tau^*$. The risky asset pays off at time t_{pay}, which also depends on the realization of $\overline{\overline{F}}$ and hence on t_0.