Working Papers

FEDERAL RESERVE BANK OF BOSTON

No. 20-11

Fiscal Expansions in the Era of Low Real Interest Rates

Vaishali Garga

Abstract:
Low natural real interest rates limit the power of monetary policy to revive the economy due to the zero lower bound (ZLB) on the nominal interest rate. Fiscal stabilization via higher government debt is frequently recommended as a policy to raise the natural real interest rate. This paper builds a non-Ricardian framework to study the tradeoffs associated with a debt-financed fiscal expansion and show that even in a low real interest rate environment, higher debt doesn't necessarily raise the real interest rate. The effect of the expansion is non-monotonic: Increasing debt raises the natural real interest rate at low levels of debt, while at high levels it perversely lowers the natural real interest rate. This threshold level of debt, beyond which the effect becomes perverse, is a function of the expected duration of the low interest rate state. In a calibrated 60-period quantitative life cycle model with aggregate uncertainty, if the low state is expected to last two years, this threshold level of debt is 250 percent of the GDP. The insights from this paper are directly applicable to a ZLB episode in a model with nominal frictions.

JEL Classifications: E52, E62, E63, D84
Keywords: debt, fiscal policy, expectation, uncertainty

Vaishali Garga is an economist in the research department of the Federal Reserve Bank of Boston. Her email is vaishali.garga@bos.frb.org.

The author owes invaluable debt to Gauti Eggertsson and David Weil for continuous advice and extensive comments on various versions of this paper. She thanks Joaquin Blaum, Gregory Casey, Violeta Gutkowski, Mrityunjay Kothari, Alessandro Lin, Neil Mehrotra, Pascal Michaillat, María Luengo-Prado, Jenny Tang, Jacob Robbins, and seminar participants at various universities and conferences for helpful comments and suggestions.

This paper previously circulated under the title "Fiscal Expansions in Secular Stagnation: What If It Isn't Secular?"

This version: April 2020 **https://doi.org/10.29412/res.wp.2020.11**

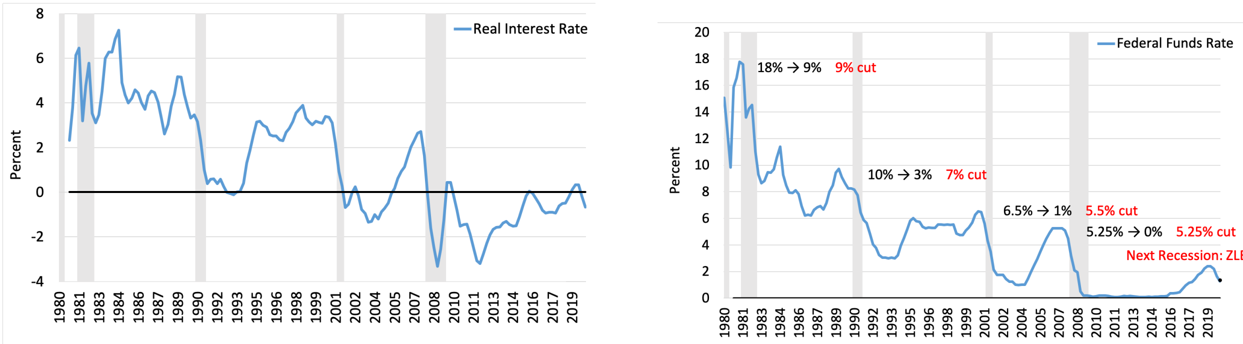
1 Introduction

Real interest rates in the United States have declined persistently over the past three decades (see Figure 1a). A similar trend emerges when one looks at the yield on a 10-year Treasury Inflation-Protected Security (TIPS) of constant maturity. This has led to concerns that the economy may have entered an era of low real interest rates with no force toward their recovery.

Low real rates complicate the use of monetary policy to fight recessions due to the zero lower bound (ZLB) on the nominal interest rate. This is because the central bank's main policy instrument is the nominal interest rate, which it cuts during recessions in order to stimulate the economy. The nominal interest rate is, on average, equal to the real rate plus the central bank's inflation target. When real rates are low in normal times, the average nominal interest rate in normal times is correspondingly lower, thereby giving monetary policy less room to reduce it in the event of a recession. Figure 1b shows the cut in the federal funds rate (FFR) in response to the last four recessions as identified by the National Bureau of Economic Research (gray bars in the figure). The Fed had room to cut the nominal rate from 18 percent to 9 percent (900 basis points) in the early 1980s, from 10 percent to 3 percent (700 basis points) in the early 1990s, from 6.5 percent to 1 percent (550 basis points) in the early 2000s, and from 5.25 percent to 0 percent (525 basis points) in 2008. As of March 2020 (before the COVID-19-pandemic), the most the Fed could cut the FFR was 25 basis points. The decline in the nominal rate over the first part of this period was due to declining inflation expectations. However, the most recent decline in the nominal rate was driven by the decline in the real rate, which remained around 0 percent (as of March 2020) even as the US economy approached full employment.

Many suggest that if low real rates represent a permanent change—a "new normal"—then fiscal policy is needed to restore the stabilizing capacity of monetary policy. Higher debt in most models, as long as they are not fully Ricardian, directly increases the real rate of

Figure 1: United States: Real interest rate and federal funds rate



(a) Real Interest Rate

(b) Federal Funds Rate Cuts

Source: FRED.
Notes: This figure shows the time series of the real interest rate in the United States (panel [a]) and the federal funds rate cuts in subsequent recessions in the United States (panel [b]). The real interest rate is calculated as the difference between the federal funds rate and expected inflation. Expected inflation is computed as a four-quarter moving average of actual inflation.

interest, which in turn gives monetary policy more room.[1]

The question this paper asks is a simple one: Does increasing government debt unambiguously increase the market-clearing real interest rate in the economy? The answer is, not always. There are two potential problems with the strategy of raising debt, raising the real interest rate, and thus getting out of this era of persistently low real rates. First, it may be that the low real rates are not a permanent phenomenon but are just transitory—and that if left alone, real rates will normalize in the future. If this is true, and we issue all this debt now, we will accumulate a lot of debt, which will have to be financed at positive real rates. The second problem is that if people anticipate the first problem, then debt may not have the desired effect even today, when real interest rates are low. This is because in the state of the world where real interest rates have normalized, accumulated debt will impose costs on the economy by triggering forces that reduce output (distortionary taxes, for example). The expectation of these costs may then induce further savings against this potential state and reduce the positive effect of debt even today. If these costs are severe

[1] An alternative is to increase the central bank's inflation target, which reduces the chances of hitting the ZLB in the future. However, it is not without costs, such as increased relative price dispersion and inflation volatility (Coibion, Gorodnichenko and Wieland (2012)), and also risks jeopardizing the credibility of monetary policy, which has worked hard to establish a credible nominal anchor of 2 percent.

enough, the precautionary savings generated by higher debt may be so large that the real interest rate falls in equilibrium. This paper formalizes the preceding argument.

Consider an environment in which real interest rates are low (called the low state) but are driven by a two-state Markov process, so that they may rise with a certain probability in every period (the state where they have risen is called the business-as-usual state). The model has overlapping generations, so that fiscal policy can have nontrivial non-Ricardian effects, and taxes are distortionary. There are two main findings. One, the effect of increasing debt depends on the probability of switching in the two-state Markov process; that is, it depends on the probability of reverting to the business-as-usual state. If the probability of reversion is low, then the regular intuition that increasing debt will increase the real interest rates is going to apply; but if the probability of reversion is high, then increasing debt will—perversely—lower the real interest rates. Two, the effect of debt is non-monotonic in the initial debt-to-GDP ratio. Increasing debt increases the real interest rates at low initial debt-GDP ratios, but at high initial debt-to-GDP ratios it perversely decreases the real interest rates. The threshold level of debt, beyond which the effect becomes perverse, is a function of the probability of switching in the two-state Markov process (which can be mapped into an expected duration of the low state). The key point here is that agents' expectations about the possibility of a future business-as-usual state undo the regular intuition that higher debt leads to higher real interest rates.

The mechanism relies on output distortions created by debt in the business-as-usual state, where the real interest rates have normalized. Expectation of this future contingency where disposable income is lower feeds into the current savings decision of forward-looking agents and induces them to increase the supply of loanable funds more than the initial increase in the demand for these funds (created by higher debt). These output distortions arise due to distortionary effects of income taxation on the labor supply and capital accumulation decisions of agents. They are further aggravated by the possibility of a sovereign default, which imposes a risk premium on government bonds, and an output loss if the default is realized.

In a calibrated 60-period quantitative life cycle model, with aggregate uncertainty regarding the path of future real interest rates (driven by a two-state Markov process), I find that if the expected duration of the low state is 1.16 years (or shorter), then increasing debt beyond its current level of 110 percent of GDP will perversely reduce the natural real interest rate. If the expected duration is longer than 1.16 years, then there is room to use debt policy to fight the prevailing low natural real interest rate. How much room there is depends on the expected duration of the low state. If the expected duration is two years, for example, then increasing debt to as much as 250 percent of GDP will increase the natural real interest rate, but further increases in debt will have a perverse effect. If the expected duration is 10 years, then there is room to increase debt to about 500 percent of GDP.

I focus the entire analysis on a real model, with the understanding that with the addition of nominal frictions, the perverse effect of debt in terms of the market-clearing real interest rate will translate to a contractionary effect on output.[2] In particular, if the low state is the one where the ZLB binds, then increasing debt will worsen the output gap in the economy if either the expected duration of the low state is short or the initial debt-to-GDP ratio is high.

Related Literature

This paper contributes to several existing literatures. First, it extends our understanding of the effectiveness of public debt as a stabilization policy tool in a low real interest rate environment. The recent literature on secular stagnation proposes an increase in public debt as a policy response to the problem of persistently low real rates (Eggertsson and Mehrotra (2014); Summers (2015); Kocherlakota (2015)). In a quantitative life cycle model, Eggertsson, Mehrotra and Robbins (2019) show that debt would need to nearly double from

[2]In a model with nominal rigidities, if the market-clearing rate is sufficiently negative (lower than negative of the inflation target), then the market real rate becomes stuck above it due to the ZLB on the nominal interest rate. This wedge then shows up as an output gap in the economy. If the government debt lowers the market-clearing real rate even further, it increases this wedge even more, which leads to a larger output gap in the economy.

118 percent to 215 percent of GDP to increase the real rate from −1.47 percent to 1 percent. Such a large increase in debt, however, presents risks to the economy due to the possibility that real interest rates may rise in the future. Using data from 1870 through 2013 for advanced economies, Mehrotra (2017) finds that despite current conditions of $r < g$, there is a moderate probability of reversion to conditions with $r > g$ over a 5- or 10-year horizon. A sharp rise in interest rates can quickly worsen debt dynamics and trigger the need for a fiscal consolidation. The model in this paper accounts for these risks and the tradeoffs created by them to show that greater public debt may not increase the natural real interest rate even in a low interest rate environment.

Second, this paper complements the New Keynesian dynamic stochastic general equilibrium literature on the effectiveness of fiscal policy at the ZLB. Several papers show that the government spending multiplier is greater than 1 when the ZLB binds (Christiano, Eichenbaum and Rebelo (2011); Woodford (2011); Eggertsson (2011); Eggertsson and Garga (2019)). Denes, Eggertsson and Gilbukh (2013) qualify this effect and show that even when interest rates are 0 percent, running budget deficits may not necessarily be expansionary. The final effect depends on the interaction of current fiscal policy with expectations about long-run taxes and spending. In a similar spirit, Erceg and Lindé (2014) show that the size of the fiscal multiplier depends negatively on the size of the fiscal stimulus, as the duration of the binding ZLB is endogenous to the stimulus size. However, all of this analysis is from a class of models in which the steady state real interest rate is pinned down by the discount factor of the savers and cannot be negative.

Finally, this paper complements the papers in two other related literatures: (1) studies on expansionary fiscal austerity, including Alesina and Ardagna (1998), Guajardo, Leigh and Pescatori (2014), Fatás and Summers (2018); and (2) studies on initial-debt-dependent effects of fiscal policy, including Reinhart and Rogoff (2010), Perotti (1999), Nickel and Tudyka (2013). The complementarity arises for two reasons. First, while these literatures are more about the endogeneity of interest rates/spreads when debt is high (critics of these papers often cite low interest rates as a reason to not worry about these perverse effects), my paper demonstrates the possibility of a low interest rate environment also being subject to the these perverse effects. Second, the papers in these literatures are mostly empirical, while my paper formalizes the argument through a model to derive quantitative policy implications, including a threshold debt-to-GDP ratio beyond which a debt-financed fiscal expansion becomes contractionary.

The rest of the paper is organized as follows. Section 2 begins with a two-generation overlapping generations model with an exogenous output cost of debt for an analytical derivation of the main results regarding the perverse effects of a debt-financed fiscal expansion. Section 3 extends this model by including endogenous labor supply and capital accumulation decisions. Section 4 derives the quantitative policy implications through the lens of a calibrated life cycle model with aggregate uncertainty. Section 5 concludes.

2 Illustrative Model: Exogenous Output Cost of Debt

In this section, we build a theoretical model to illustrate the *perverse* effect of a debt-financed fiscal expansion, that is, to illustrate the conditions under which an increase in government debt can reduce the market-clearing real interest rate in a frictionless environment.

Before delving into the details of the model, it is useful to clarify the terminology that will be used in the rest of the paper.

Definition 2.1. Benchmark Real Interest Rate: The market-clearing real interest rate in the frictionless (flexible price/wage) economy in the absence of any fiscal policy intervention, that is, when fiscal policy is at its baseline steady state.

Definition 2.2. Natural Real Interest Rate: The market-clearing real interest rate in the frictionless economy accounting for fiscal policy intervention, that is, when fiscal policy deviates from its baseline steady state.[3]

[3]In the absence of a fiscal policy change, the natural real interest rate coincides with the benchmark real interest rate.

We begin with a two-generation overlapping generations (OLG) model. There are three main stakeholders: households, firms, and the government. There are two key differences relative to the common exposition of OLG models: (1) a stochastic process driving the benchmark real interest rate in the economy and (2) a careful specification of fiscal policy. The stochastic process is such that in the absence of any policy intervention, the benchmark real interest rate is negative in the current period but reverts to positive territory with a positive probability in every ensuing period. Once it becomes positive, it stays at that level forever (assumption 1). On the fiscal policy front, the government chooses an exogenous level of spending and stock of one-period risk-free debt, while taxes are determined endogenously to balance the government's budget constraint in every period (assumption 2).

Households

The household lives for two periods/generations, middle age and old age, and maximizes utility from the consumption of one aggregate good according to the following utility function:

$$
\begin{aligned}
U_t(c_t^m, c_{t+1}^o) = \max_{c_t^m, c_{t+1}^o} \mathbb{E}_t \left\{ \frac{(c_t^m)^{1-\gamma}}{1-\gamma} + \beta D_t \frac{(c_{t+1}^o)^{1-\gamma}}{1-\gamma} \right\} \\
\text{s.t.} \quad c_t^m = w_t L_t^m + \theta_z Z_t - T_t^m - a_t^m, \\
c_{t+1}^o = w_{t+1} L_{t+1}^o + (1-\theta_z) Z_{t+1} - T_{t+1}^o + (1+r_t) a_t^m.
\end{aligned}
$$

Here, c_t^m is the consumption when middle-aged; c_t^o is the consumption when old; $\beta \in (0,1)$ is the discount factor of the household; $\gamma > 0$ is the coefficient of relative risk aversion; L_t^m and L_t^o are the labor endowments of the middle-aged and old, respectively; Z_t is the profit from firm ownership; T_t^m and T_t^o are the lump-sum taxes imposed by the government on the middle-aged and old, respectively; and a_t^m consitutes the savings of the middle-aged. I assume an exogenous division of the labor endowment between the two generations governed by the parameter θ; that is, $L_t^m = \theta L$ and $L_t^o = (1-\theta)L$, and the intergenerational profit distribution is governed by θ_z. D_t is an inter-temporal wedge shock. This shock plays a crucial role in the analysis, as will be explained shortly.[4]

[4]The inter-temporal wedge D_t can be microfounded using forces analyzed in Eggertsson, Mehrotra and

The inter-temporal consumption decision of the middle-aged households is given by their Euler equation:

$$(c_t^m)^{-\gamma} = \beta D_t(1+r_t)\mathbb{E}_t \left(c_{t+1}^o\right)^{-\gamma}.$$

They like to smooth their consumption across the two periods of their life. This smoothing motive is governed by their effective discount factor, βD_t; the real interest rate, r_t; and their coefficient of relative risk aversion, γ. This is an important equation in the model, as expectations will matter for the economy via their effect on the consumption of the middle-aged (savers).

The consumption of the old is given by their budget constraint.[5]

Firms

Firms use labor to produce the final output in the economy according to the following production function:

$$Y_t = L_t^{1-\alpha}.$$

When $\alpha = 0$, there are constant returns to scale in production. In this case, profits are zero, and the ownership structure of the firm is irrelevant for the results. However, if $\alpha \in (0, 1)$, then there are decreasing returns to scale, and here I assume that the division of firm ownership between the middle-aged and the old is governed by the parameter θ_z.

Firms choose labor to maximize profits, taking wages and prices as given:

$$\max_{L_t} Z_t = P_t Y_t - W_t L_t.$$

This yields the inverse demand function for labor in the economy.

Government

The government issues one-period risk-free bonds (B_t^g) and collects lump-sum taxes (T_t) in every period to finance its existing stock of debt (B_{t-1}^g) plus purchases (G_t). While

Robbins (2019) such as a deleveraging shock, slowdown in productivity growth, relative price of investment, inequality, etc. The main feature of the shock that is significant for the current analysis is that there is uncertainty about its future path.

[5] I assume that households are born and die without assets.

purchases and debt are exogenous, the taxes are determined endogenously to maintain a balanced budget in every period:[6]

$$B_t^g + T_t^m + T_t^o = (1 + r_{t-1})B_{t-1}^g + G_t.$$

2.1 Equilibrium under Flexible Prices

Now we can use this setup to analyze the effect of a debt-financed fiscal expansion on the economy. The economy is in equilibrium when the goods market clears, which is also equivalent to the asset market clearing. This requires aggregate demand to equal aggregate supply. Aggregate demand is the sum of the consumption of the two generations plus government purchases. Aggregate supply depends on labor supply. Since households do not obtain any disutility from working, they are willing to supply all their labor endowment at any positive wage. Thus, under flexible prices, the economy is always at full employment. Any disequilibrium created by exogenous shocks is equilibriated through adjustments in the real interest rate.

The flexible-price competitive equilibrium in the economy is defined as a sequence of endogenous variables $\{Y_t, c_t^m, c_t^o, L_t, w_t, T_t^m, T_t^o, Z_t, r_t, a_t^m\}$, so that given prices and wages, households maximize utility, firms maximize profits, the government balances its budget, and the aggregate goods/asset market clears for a given exogenous path of $\{B_t^g, G_t, D_t\}$ and labor endowment $\{\bar{L}\}$. The full equilibrium system is presented in Appendix A.1.

2.2 Characterization of Equilibrium

There are two possible states of the economy: low (L) state and business-as-usual (B) state, driven by the inter-temporal wedge shock D_t. This wedge takes on two values: D_L and D_B. D_B is normalized to 1, while $D_L > D_B$. A higher D_t implies that households derive a higher

[6]Assuming proportional income taxes yields similar results; due to the exogeneity of labor supply, proportional income taxes are non-distortionary. The benefit of assuming lump-sum taxes is that it simplifies the algebra for analytical tractability.

utility from the same amount of future consumption. This induces them to save more from any given income, thereby reducing the benchmark real interest rate. This is the low state, which corresponds to D_L. On the other hand, a low D_t reduces savings and increases the benchmark real interest rate. This is the business-as-usual state, corresponding to D_B.

To solve the model, I make use of a simple assumption that is based on Woodford and Eggertsson (2003) and is now common in the New Keynesian literature.

Assumption 1. *Inter-temporal Wedge Process: The economy is in a constant low-state solution, with $D_t = D_L$. In every period, there is a probability $(1-\mu)$ that D_t will revert to its business-as-usual state value D_B. Once the economy is in the business-as-usual state, it stays there forever. The stochastic period in which the shock reverts to its business-as-usual state value is denoted by Γ.*

Additionally, for fiscal policy, I assume that government purchases are constant, the government targets a constant stock of end-of-period/interest-inclusive debt,[7] and the resulting tax burden is shared by the two generations as governed by the parameter θ_τ.

Assumption 2. *Fiscal Policy:*

$$G_t = \bar{G}, \quad \forall t,$$

$$\mathbb{W}_t = (1+r_t)B_t^g = \mathbb{W}^*, \quad \forall t,$$

$$T_t^m = \theta_\tau \left[(1+r_{t-1})B_{t-1}^g - B_t^g + G\right],$$

$$T_t^o = (1-\theta_\tau) \left[(1+r_{t-1})B_{t-1}^g - B_t^g + G\right].$$

Under assumption 1, the benchmark real interest rate in the current low state is negative (with a high D_L) but reverts to being positive with a positive probability $(1-\mu)$ in every ensuing period. This means that government debt can be issued at a negative interest rate today but would need to be financed at a higher positive interest rate if the economy

[7] This is an assumption made for the sake of analytical tractability—it helps to eliminate the endogenous state variable (r_{t-1}) from the system. We will work with a beginning-of-period debt target in the later sections of the paper.

reverts to the business-as-usual state. The proceeds from debt issuance are rebated to the households as tax cuts today, but there is a probability of higher taxes in the future if the benchmark real interest rate recovers. While I assume away any distortionary effects of taxation by assuming lump-sum taxes, this is not the case in the real world, where debt service payments are financed through distortionary income taxes. For now, I assume a reduced form for this distortion.

Assumption 3. *Exogenous Output Cost of Debt: Government debt is costly in terms of output in the business-as-usual state.*[8] *This cost is of the form:*

$$Y_t = L_t^{1-\alpha} - \nu_g \mathbb{W}_t,$$

where ν_g determines the cost of output per unit debt.

This cost arises endogenously in models with endogenous labor supply and capital accumulation decisions via distortionary taxation and crowding out, and in models with sovereign default risk. These elements will be added to the illustrative model in Section 3.

Under assumptions 1, 2, and 3, the optimality conditions of the model can be simplified and reduced to one equation, which is the Euler equation, in one endogenous variable $\{r_t\}$, given exogenous $\{\mathbb{W}_t, D_t, \bar{G}\}$ and parameter set $\{\Theta, \nu_g, \bar{G}\}$, where $\Theta \equiv \{\beta, \gamma, \alpha, \bar{L}, \theta_z, \theta_\tau\}$ (details in A.1).

2.3 Policy Experiment: Debt-financed Fiscal Expansion

To understand the effect of a debt-financed fiscal expansion in an economy where the benchmark real interest rate is negative but may become positive in the future, I contemplate the following experiment: Suppose the economy begins in the low state with a stock of

[8] In the model where taxes are distortionary, higher debt will have output effects in the low state as well. In particular, the proceeds from higher debt will lead to lower tax rates, which will increase output by crowding in labor supply/capital. Accounting for this positive effect does not change the qualitative results, but it complicates the analytical expressions. So, I do not account for it in the illustrative model, but it will be accounted for in the baseline and quantitative models in the following sections.

government debt equal to $\mathbb{W}^*_{orig}$, government purchases equal to $\bar{G}$, and taxes determined endogenously according to assumption 2. Now suppose that the government unexpectedly announces an increase in its stock of debt to $\mathbb{W}^*_{new}$. At negative rates, this debt issuance corresponds to a one-time tax cut for the households (as I assume that government purchases do not change). The government also announces that it will permanently maintain debt at this new high level.

To analyze the effect on the natural real interest rate, it is helpful to first abstract from the transition dynamics and focus on the *constant solution* to establish analytical results. The constant solution involves comparing the low-state natural real interest rate in the steady state of the economy with a low ($\mathbb{W}^*_{orig}$) versus high ($\mathbb{W}^*_{new}$) stock of debt. In other words, it is equivalent to doing comparative statics with respect to the stock of debt inherited from the past. The following two propositions characterize this constant solution.

Proposition 1. *Suppose assumptions 1, 2, and 3 hold.*
If $D = D_B$, then the economy is in the business-as-usual state with a constant solution for the natural real interest rate given by:

$$r_B = f\left(\mathbb{W}^*, \nu_g, D_B; \Theta\right).$$

If $D = D_L$, then the economy is in the low state with a constant solution for the natural real interest rate given by:

$$r_L = g\left(\mathbb{W}^*, D_L, \mu, \nu_g, D_B; \Theta\right).$$

Proof. The proof, along with closed-form expressions, is provided in Appendix A.2. □

Taking the partial derivative of the constant low-state natural real interest rate (r_L) with respect to the stock of government debt ($\mathbb{W}^*$) yields the next proposition, which characterizes the effect of expansionary fiscal policy on the natural real interest rate.

Proposition 2. *Under assumptions 1, 2, and 3, the effect of increasing government debt on the low-state constant natural real interest rate is given by:*

$$\frac{dr_L}{\mathbb{W}^*} = \mu\left(\frac{\partial r_L}{\partial B^g}\right)^{direct} + (1-\mu)\times\left(\frac{\partial r_L}{\partial B^g}\right)^{indirect},$$
$$\textit{where, } \left(\frac{\partial r_L}{\partial B^g}\right)^{direct} = i\left(D_L, \mathbb{W}^*, \nu_g, D_B; \Theta\right) \geq 0,$$
$$\textit{and, } \left(\frac{\partial r_L}{\partial B^g}\right)^{indirect} = i\left(D_L, \mathbb{W}^*, \nu_g, D_B, \frac{dr_B}{d\mathbb{W}^*}; \Theta\right) \leq 0.$$

Proof. The proof, along with closed-form expressions, is provided in A.3. □

In the low state, the effect of increasing debt on the natural real interest rate is driven by two opposing forces: a direct effect, which is positive, and an indirect effect, which is negative. The direct effect captures the effect if the economy were to stay in the low state in the next period, while the indirect effect captures the effect from the possible exit from the low state and entrance into the business-as-usual state. Conditional on $\mathbb{W}^*$, the strength of the indirect effect depends on the probability of the economy reverting to the business-as-usual state in the next period, that is, $(1-\mu)$. Given the two opposing forces driving the overall effect, a threshold probability emerges; if the probability of reversion is higher than this threshold, then the indirect effect via expectations dominates, and increasing debt has a net negative (perverse) effect on the natural real interest rate. On the other hand, if the probability of reversion is lower than this threshold, then increasing debt raises the natural real interest rate. The following proposition characterizes this threshold.

Proposition 3. *Suppose assumptions 1, 2, and 3 hold and, $D = D_L$. Then:*

$$\exists \mu^* = \mu^*\left(r_L, D_L, \nu, \Theta\right) \geq 0, \ \textit{such that} \begin{cases} \left(\frac{\partial r_L}{\partial B^g}\right) < 0, \textit{if } \mu < \mu^* \\ \left(\frac{\partial r_L}{\partial B^g}\right) \geq 0, \textit{if } \mu \geq \mu^*. \end{cases}$$

Proof. The proof, along with closed-form expressions, is provided in Appendix A.4. □

A direct corollary of this proposition is that when other parameters are held fixed, the threshold μ^* is an increasing function of ν_g.

Corollary 1. *Under assumptions 1, 2, and 3, when $D = D_L$:*

$$\frac{d\mu^*}{d\nu_g} \geq 0.$$

Proof. The proof is provided in A.5. ☐

The logic of all the propositions will be discussed in the numerical examples in the next subsection, where I will also show an analog of proposition 3 for $\mathbb{W}^*$. In particular, I will show that, conditional on μ, there exists a threshold $(\mathbb{W}^*)^*$ such that increasing debt to this threshold has a positive effect on the natural real interest rate, but further increases in debt beyond this threshold have a perverse effect.

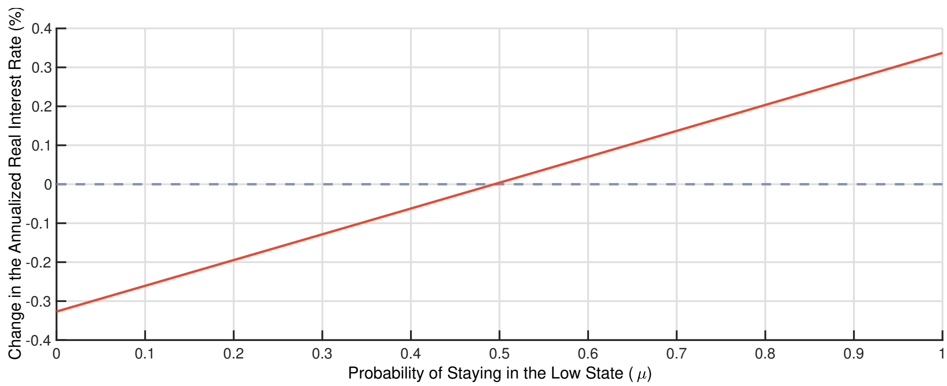
2.4 Numerical Results

At this stage, it is useful to attach illustrative parameters to the model in order to visualize the closed-form results derived so far and understand the logic behind them. The details of the parameterization are provided in Table 4 in Appendix A.6.

Result 1: The effect of fiscal expansion depends on μ.

Figure 2 shows the numerical analog of proposition 3. The effect of an increase in debt on the low-state natural real interest rate (constant solution) depends positively and monotonically on the probability of being in the low state tomorrow (μ). At one extreme, if $\mu = 1$, that is, the economy stays in the low state forever, increasing debt unambiguously increases the natural real interest rate. $\mu = 1$ is the implicit assumption in the "secular" stagnation literature, thereby rationalizing its policy recommendation of fiscal stabilization via higher public debt to revive the natural real interest rate. At the other extreme, if $\mu = 0$, that is, the economy reverts to the business-as-usual state with certainty in the next period, then increasing debt unambiguously has a perverse effect on the natural real interest rate. In between these extremes lies $\mu^* \approx 0.5$, where the sign of the effect switches.

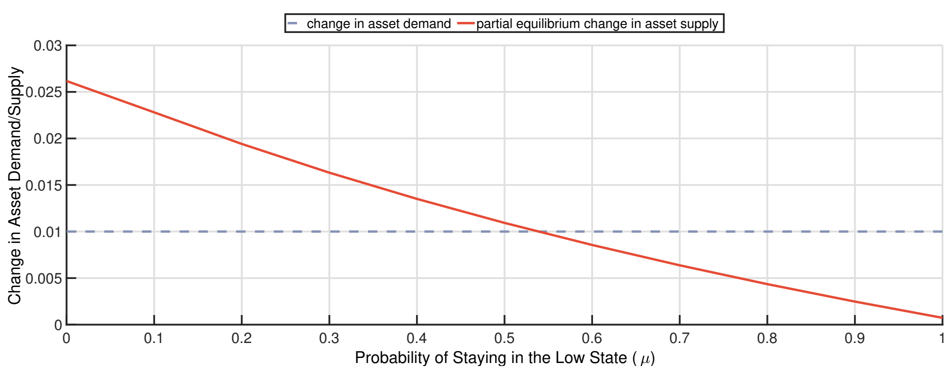
Figure 2: Effect of increasing debt on the low-state natural real interest rate



Source: Author's calculations.
Note: This figure plots the change in the annualized real interest rate (in percent) against different probabilities of staying in the low state in the next period, ranging from 0 to 1. The entire computation holds the initial annual debt-to-GDP ratio fixed at 3 percent.

The intuition behind this result can be understood by analyzing the impact of the considered policy experiment on the market for loanable funds. An increase in government debt constitutes an increase in the demand for loanable funds (a rightward shift in the demand curve). The final effect on the equilibrium real interest rate depends on whether the supply for loanable funds increases by more or less than this initial increase in the demand for loanable funds (whether the supply curve shifts out by more or less than the demand curve does). The middle-aged households are the suppliers of loanable funds in this model. Their savings decision, in turn, is based on their Euler equation: $(c_L^m)^{-\gamma} = \beta D_L(1+r_L)\mathbb{E}_t\left(c_{t+1}^o\right)^{-\gamma}$, where $\mathbb{E}_t\left(c_{t+1}^o\right)^{-\gamma} = \left[\mu\left(c_L^o\right)^{-\gamma} + (1-\mu)\left(c_B^o\right)^{-\gamma}\right]$, in accordance with assumption 1. In the business-as-usual state, higher debt implies lower disposable income, and hence, lower consumption by the old. The weight on the business-as-usual state consumption-when-old in the expected consumption of the middle-aged is given by $(1-\mu)$, which is the probability of exit from the low state. When μ is smaller, this probability of exit is larger, which means that the expected consumption is lower (or the expected marginal utility of consumption is higher). This makes the middle-aged households consume less and save more today. On the other hand, when μ is higher, the probability of exit is smaller, which means that the expected consumption is higher. As a result, the middle-aged households consume more and save less today. To visualize the magnitude of this "shift" in asset supply for the entire range of μ, we can compute the (partial equilibrium) change in the supply of loanable funds. In particular, we can compute the new asset supply choice of the middle-aged households, holding fixed the real interest rate at its initial equilibrium value but accounting for the increase in government debt to its new level. Figure 3 plots these changes. We can see that when μ is lower, that is, when the households believe that the probability of the economy transitioning to the business-as-usual state in the next period is sufficiently high, the change in asset supply is greater than the change in asset demand, which ultimately results in a lower natural real interest rate in the new equilibrium. On the other hand, when μ is higher, the change in asset supply is smaller than the change in asset demand, which ultimately results in a higher natural real interest rate in the new equilibrium.

Figure 3: Effect of increasing debt on the loanable funds market

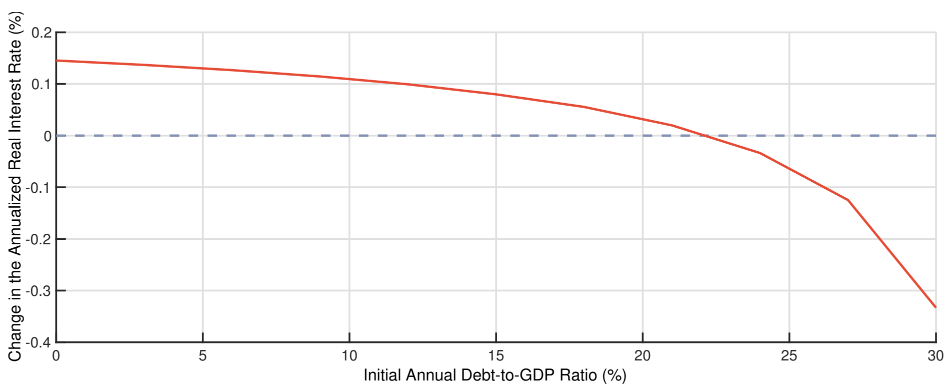


Source: Author's calculations.
Notes: This figure plots the change in asset demand and the partial equilibrium change in asset supply against different probabilities of staying in the low state in the next period, ranging from 0 to 1. In computing the partial equilibrium asset supply, the real interest rate is artificially held fixed at its level before debt was increased. The entire computation holds the initial annual debt-to-GDP ratio fixed at 3 percent.

Result 2: The effect of fiscal expansion depends on the initial debt-to-GDP ratio.

Figure 4 illustrates the analog of Proposition 3 with respect to $\mathbb{W}^*$. It shows the non-monotonicity in the response of the low-state natural real interest rate with respect to the initial debt-to-GDP ratio. If the economy begins at a low initial debt-to-GDP ratio, increasing debt increases the natural real interest rate; however, for an economy with an already high debt-to-GDP ratio, increasing debt has a perverse effect on the natural real interest rate. In other words, for any given probability of staying in the low state tomorrow (μ), increasing debt first has a positive effect on the low-state natural real interest rate up to a threshold. Any further increases in government debt beyond this threshold would have a perverse effect on the low-state natural real interest rate.

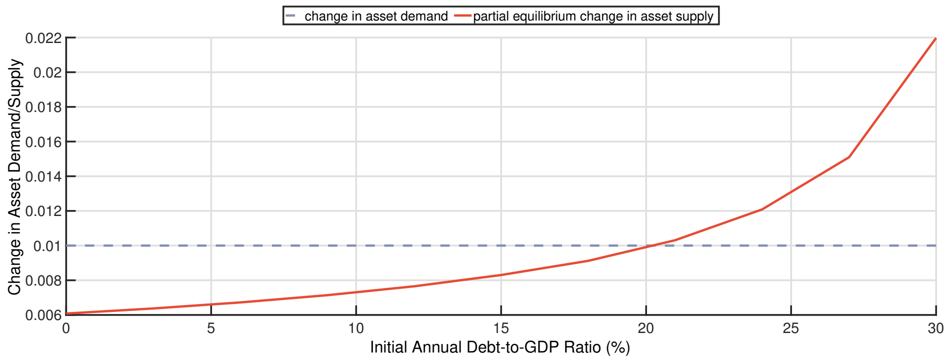
Figure 4: Non-monotonicity of change in natural real interest rate with respect to the initial debt-to-GDP ratio



Source: Author's calculations.
Notes: This figure plots the change in the annualized real interest rate (in percent) against different initial debt-to-GDP (annual) ratios. The computation holds the probability μ, in the stochastic process for the inter-temporal wedge shock, fixed at 0.7.

Again, the intuition behind this result can be understood by analyzing the impact of the considered policy experiment on the market for loanable funds. The change in asset demand is the same across all initial debt-to-GDP ratios by construction. What remains to be studied is the partial equilibrium change in asset supply for these different initial debt-to-GDP ratios. Figure 5 plots these changes in the asset market. For low initial debt-to-GDP ratios, the partial equilibrium change in asset supply is dominated by the change in asset demand, which results in a net positive effect of the policy on the natural real interest rate. Then the debt-to-GDP ratio reaches a threshold beyond which further increases in this ratio imply larger changes in asset supply relative to asset demand, which results in a perverse effect of the policy on the natural real interest rate.

Figure 5: Non-monotonicity of the partial equilibrium change in asset supply with respect to the initial debt-to-GDP ratio



Source: Author's calculations.
Notes: This figure plots the change in asset demand and the partial equilibrium change in asset supply against different initial debt-to-GDP (annual) ratios. In computing the partial equilibrium asset supply, the real interest rate is artificially held fixed at its level before debt was increased. All of this computation holds the probability μ, in the stochastic process for the inter-temporal wedge shock, fixed at 0.7.

Sensitivity Analysis with Respect to ν_g

An important parameter in the current analysis is ν_g, which pins down how costly government debt is in the business-as-usual state. The larger this parameter, the "worse" the business-as-usual state is in terms of expected consumption of the current middle-aged. Importantly, this expectation is worse for any given probability of exit from the low state. Thus, for any given μ, as ν_g increases, the middle-aged want to save more and consume less from any given current income (based on their Euler equation). This leads to a reduction in the real interest rate for any given μ.[9] In other words, as ν_g increases, increasing debt has a perverse effect for a larger range of μ values, as shown in Table 1:

Table 1: Sensitivity Analysis with Respect to ν_g

ν_g	1	1.5	2	2.5	3
μ^*	0	0.65	0.8	0.85	0.9

[9] A graphical representation of this fact is shown in Figure 11 in Appendix A.5.

3 Extension: Endogenous Labor and Capital

So far we have seen that a debt-financed fiscal expansion can—perversely—lower the natural real interest rate if it triggers expectations of sufficiently low disposable income in the future. These negative expectations are built into the model via an assumption that debt is associated with an exogenous cost in terms of output in the business-as-usual state. In this section, we relax this assumption and introduce endogenous labor supply and capital accumulation decisions, along with income taxation, into the model. In this setting, the output cost of debt arises endogenously, and we obtain similar results.

The following equations highlight the main differences relative to the illustrative model that are relevant for the main results. The full model, complete with the optimality conditions, is presented in B.

Households maximize utility according to the following function:

$$U_t(c_t^m, c_{t+1}^o, h_t^m, h_{t+1}^o) = \max_{c_t^m, c_{t+1}^o, h_t^m, h_{t+1}^o, a_t^m} \mathbb{E}_t \left\{ \frac{(c_t^m)^{1-\gamma}}{1-\gamma} + \beta D_t \frac{(c_{t+1}^o)^{1-\gamma}}{1-\gamma} - \chi \frac{(h_t^m)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} - \chi \beta D_t \frac{(h_{t+1}^o)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\}$$

$$\text{s.t.} \quad c_t^m = w_t h_t^m (1-\tau_t) - a_t^m - K_t^s + r_t^k (1-\tau_t) K_t^s,$$

$$c_{t+1}^o = w_{t+1} h_{t+1}^o (1-\tau_{t+1}) + (1+r_t) a_t^m + (1-\delta) K_t^s.$$

Here, h_t^m and h_{t+1}^o are the labor supplies of the middle-aged and old, respectively; η is the Frisch elasticity of labor; τ_t is the income tax rate; a_t^m is the supply of loanable funds toward government bonds; K_t^s is the supply of loanable funds toward productive capital; r_t^k is the rental rate on capital; and δ is the capital depreciation rate. The middle-aged rent out the capital in the same period when it is produced; this is a reasonable assumption because every period/generation in the model represents 30 real-life years. They consume the undepreciated portion of this capital in the next period, when they old. As in the illustrative model, a shock to the inter-temporal wedge D_t will generate a negative benchmark real interest rate in the low state.

Firms now use both labor and capital in their production function: $Y_t = \left(K_t^d\right)^\alpha L_t^{1-\alpha}$.

The government obtains its revenue from income (rather than lump-sum) taxation on both wage and rental income: $B_t^g + \tau_t w_t L_t + \tau_t r_t^k K_t^s = (1 + r_{t-1})B_{t-1}^g + G_t$. In equilibrium, this implies $B_t^g + \tau_t Y_t = (1 + r_{t-1})B_{t-1}^g + G_t$ as $K_t^s = K_t^d$ in equilibrium.

We omit assumption 3 so that there is no longer an exogenous output cost of government debt in the business-as-usual state. Assumption 2 is modified so that the government targets a beginning-of-period (rather than end-of-period) stock of debt while still maintaining a constant level of purchases in every period. The tax rate is determined endogenously (there is no longer an arbitrary division of the total lump-sum tax burden between the two generations).

Assumption 4. *Fiscal Policy:* $G_t = \bar{G}, \quad B_t^g = B^{g*}, \quad \forall t.$

In the experiment considered, the government unexpectedly announces a permanent increase in its stock of debt from ${B^{g*}}_{old}$ to ${B^{g*}}_{new}$. Under assumptions 1 and 4 now, the results obtained are similar to those in the illustrative model.[10]

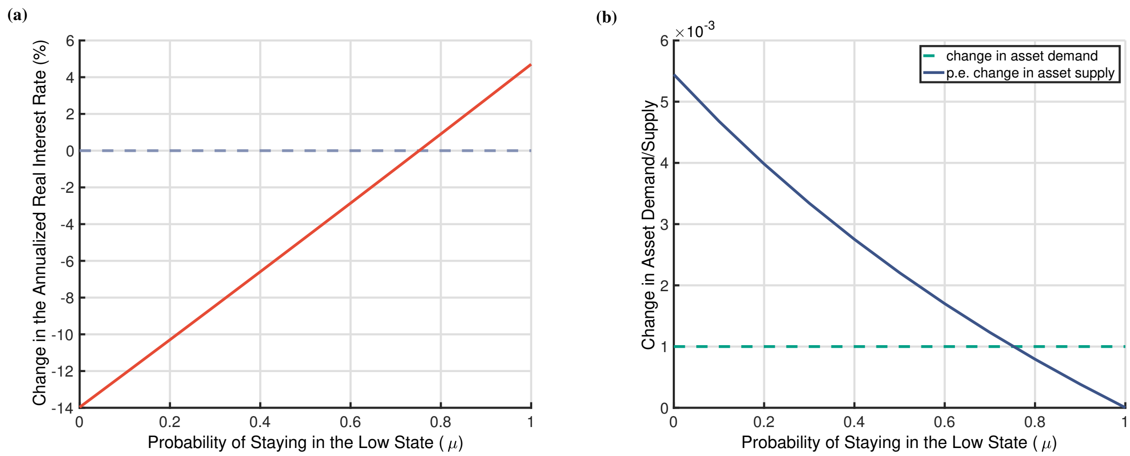
Figure 6(a) is the analog of Figure 2. The effect of debt on the low-state natural real interest rate depends on the probability of staying in the low state in the next period, μ. In particular, there is a threshold μ^*, such that for $\mu < \mu^*$, increasing debt perversely reduces the natural real interest rate, while for $\mu \geq \mu^*$, increasing debt has the desired effect of increasing the natural real interest rate. The intuition (shown in panel [b]) is also similar to the illustrative model. For $\mu < \mu^*$, the increase in supply of loanable funds toward government bonds, triggered by the expectation of being in the "bad" business-as-usual state in the next period, is so large that it dominates the initial increase in demand for these loanable funds, and the real interest rate falls in equilibrium.

Similarly, Figure 7(a) is the analog of Figure 4. A debt-financed fiscal expansion has the desired positive effect on the natural real interest rate at low initial debt-to-GDP ratios but has a perverse effect when the initial debt-to-GDP ratio is high.

The logic of these results is also analogous to the illustrative model but no longer relies on an exogenously assumed output cost of debt in the business-as-usual state. Instead, this

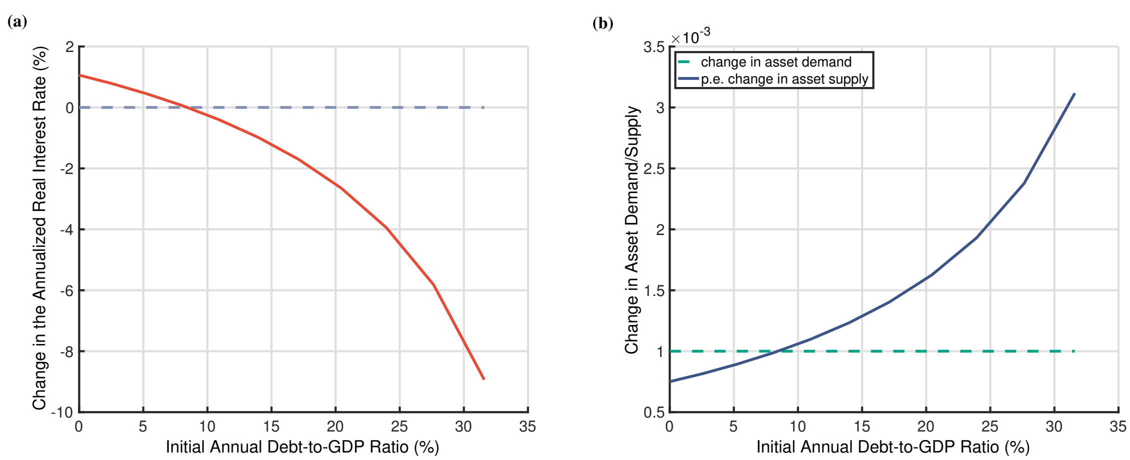
[10]Refer to Table 4 in Appendix A.6 for details of the parameterization.

Figure 6: Effect of increasing debt on the low-state natural real interest rate



Source: Author's calculations.
Notes: This figure plots the change in the annualized real interest rate (panel [a], in percent) and the change in the market for loanable funds (panel [b]) against different probabilities of staying in the low state in the next period, ranging from 0 to 1. The ratio is computed based on the annualized equilibrium output in the initial constant solution of the business-as-usual state. The computation holds the initial annual debt-to-GDP ratio fixed at 13.95 percent.

Figure 7: Non-monotonicity of change in natural real interest rate with respect to the initial debt-to-GDP ratio



Source: Author's calculations.
Notes: This figure plots the change in the annualized real interest rate (panel [a], in percent) and the change in the market for loanable funds (panel [b]), against different initial debt-to-GDP (annual) ratios. The ratio is computed based on the annualized equilibrium output in the initial constant solution of the business-as-usual state. The computation holds the probability μ, in the stochastic process for the inter-temporal wedge shock, fixed at 0.7.

cost arises endogenously because in the business-as-usual state, higher debt must be financed at a higher real interest rate (compared with the benchmark real interest rate before the policy change), which necessitates a higher income tax rate. This higher taxation reduces disposable income both by negatively distorting labor supply and through the crowding out of productive capital. A lower disposable income results in lower consumption. The consumption-smoothing middle-aged respond to this drop in expected consumption by saving more today, which then applies downward pressure on the real interest rate.

A crucial parameter in this analysis is the Frisch elasticity of labor supply, η. This elasticity determines the responsiveness of labor supply to changes in the tax rate and, hence, the distortionary effect of debt on output in the business-as-usual state. With a higher η, the same increase in the income tax rate corresponds to a larger drop in labor supply and, hence, disposable income.

This effect can be be seen in Table 2, which shows that as η increases, the implied output cost of debt in the business-as-usual state, computed as $\nu_g = -\frac{dY_B}{dB^{g*}}$, also increases. Thus, with higher η, the perverse expectations channel will be stronger for any given μ. Alternatively, the threshold μ^* increases weakly in η.[11]

Table 2: Sensitivity Analysis with Respect to η

η	1.4	1.8	2.2	2.6	3	3.4
implicit ν_g	27.936	29.61	30.858	31.825	32.595	33.224
μ^*	0.75	0.85	0.9	0.9	0.9	0.95

4 Quantitative Life Cycle Model with Aggregate Uncertainty

We now turn to a medium-scale version of the baseline model for a quantitative analysis of the effect of a debt-financed fiscal expansion in a low interest rate environment. The model is based on Heer and Maussner (2009) and Auerbach and Kotlikoff (1987).

The economy consists of a large number of households with identical utility functions.

[11] A graphical representation is shown in Figure 12 in Appendix B.

These households live for $T + R$ periods. Each generation is of measure $\frac{1}{T+R}$. In the first T periods of life, households consume and work while endogenously deciding the amount of labor to supply based on market prices. In the last R periods, they are retired and receive no labor income but receive pension payments from the government. They die with certainty at age $T + R + 1$. They maximize expected lifetime utility when they are of age 1 at time t according to the following function:

$$\mathbb{E}_t \Sigma_{s=1}^{T+R} \beta^{s-1} D_{t+s-1} u\left(c_{t+s-1}^s, n_{t+s-1}^s\right).$$

Instantaneous utility is a function of both consumption and leisure:

$$u(c, n) = \frac{c^{1-\gamma}}{1-\gamma} - \chi \frac{n^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}.$$

The working agent of age s faces the following budget constraint in period t:

$$a_{t+1}^{s+1} = (1 + r_t) a_t^s + w_t n_t^s (1 - \tau_t^s) - c_t^s, \quad s = 1, \ldots, 40.$$

The budget constraint of the retired worker is given by:

$$a_{t+1}^{s+1} = (1 + r_t) a_t^s + b - c_t^s, \quad s = 41, \ldots, 60.$$

Here, c_t^s is the consumption of age s household in period t, n_t^s is the labor supply of age s household in period t, a_t^s is the supply of loanable funds by the age s household in period t, which becomes a part of the disposable income for period $t + 1$ when the household is age $s + 1$. The households are assumed to be born without assets, and they do not leave assets behind when they die. This means $a_t^1 = a_t^{T+R+1} \equiv 0$. Additionally, in retirement, labor supply is zero, so that $n_t^{T+1} = n_t^{T+2} = \ldots = n_t^{T+R} = 0$. The pension b is constant. As in the baseline model, the future streams of utility from consumption and disutility from labor are discounted using the discount factor β and inter-temporal wedge shock D_t.

Production is characterized by constant returns to scale and assumed to be Cobb-Douglas:

$$Y_t = N_t^{1-\alpha} K_t^{\alpha}.$$

In a factor market equilibrium, factors are rewarded with their marginal product.

The government issues one-period debt in every period and collects wage income taxes to finance its purchases, pension spending, and payments on servicing the last period's debt:

$$\tau_t w_t N_t + B^g_{t+1} = (1 + r_t)B^g_t + G_t + \tfrac{R}{T+R}b.$$

The policy experiment is the same as previously. The economy begins in the low state with a negative real interest rate. The government unexpectedly announces a permanent increase in its stock of debt. The proceeds from this debt, issued at the current negative real interest rate, are rebated to the households via tax cuts (in accordance with assumption 4) . However, there is a risk of real interest rate normalization in the future (in accordance with assumption 1). In case the economy exits the low state and enters the business-as-usual state in the next period, this higher debt stock has to be serviced at the positive real interest rate that prevails in that state. The goal is to quantitatively analyze the effect of a debt-financed fiscal expansion in a low real interest rate environment.

In the presence of a stochastic process driving the benchmark real interest rate in the economy, solving for the transition dynamics in a fully nonlinear model presents a challenge. To address the aggregate uncertainty, I first reduce the system to a linearized rational expectations system. This requires solving for the non-stochastic steady state equilibrium of the nonlinear model and then linearizing the system around this equilibrium. Once the system is cast in this form, the assumptions on the shock structure allow us to solve the model transition dynamics recursively in the two states (see Eggertsson et al. (2020) for details). We start by solving for the dynamics in the business-as-usual state and work backward to the period when the system is in the low state. It is important to keep in mind that outcomes in later states influence behavior in earlier ones through expectations, while outcomes in the earlier states influence outcomes in the later state through endogenous state variables. The computation details, along with the numerical solution algorithm, are outlined in Appendix C. The algorithm is based on Klein (2000).

4.1 Calibration

We calibrate the quantitative model to the current moments in the US data and use this calibrated model to derive the effect of increasing public debt on the natural real interest rate in the low state. The calibration strategy includes the use of a set of parameters directly from the literature and another set to match moments in the data. Panel A of Table 3 shows the parameters from the literature, while Panel B shows the parameters based on the data. Fiscal variables are set to match their contemporary values. Government purchases are set at 18.2 percent of GDP, the baseline initial debt-to-GDP ratio is set at 110 percent, and the pension spending is set at 6.8 percent of GDP. The income tax rate is then determined endogenously based on the government's period budget constraint.

As with the illustrative model, an important parameter in the following analysis is the Frisch elasticity of labor supply, η. Reichling and Whalen (2012), in their literature survey paper, suggest a range of 2 to 4 for the macro Frisch elasticity.[12] Prominent examples include that of Hall (2009), whose calibration exercise estimates the macro Frisch elasticity of labor supply in a sticky-wage model at 1.9, and that of Rogerson and Wallenius (2009), whose calibration exercise in an overlapping generations model with labor taxation produces estimates ranging from 2.3 to 3. Smets and Wouters (2007) employ parameter estimation in a DSGE model to obtain a point estimate of 1.9. In the baseline calibration, I conservatively set this elasticity to 2, which is at the lower end of the estimated range. A sensitivity analysis with respect to η is provided in Table 6 in Appendix C.

Under the chosen calibration, the steady state real interest rate is 1.03 percent and the tax rate is 37.34 percent. The steady state consumption, asset, and labor supply distributions are reported in Figure 13 in Appendix C.1.

[12]The micro-elasticity is based on the intensive margin of hours worked, while the macro-elasticity also incorporates the extensive margin. The estimates for the micro-elasticity are much smaller and range from 0.2 to 0.54 (see MaCurdy (1981), Altonji (1986), and Chetty et al. (2011) for a micro-literature survey).

Table 3: Calibration Parameters

Panel A: Literature	*Symbol*	*Value*
Discount Factor	β	0.99
Coefficient of Relative Risk Aversion	γ	1
Frisch Elasticity of Labor	η	2
Disutility from Working Parameter	χ_s	1
Capital Share	α	0.3
Depreciation Rate	δ	10% p.a.
Number of Work Years	T	40
Number of Retirement Years	R	20
Panel B: Data	*Symbol*	*Value*
Government Debt (% of GDP)	B^g	110%
Government Purchases (% of GDP)	G	18.2%
Pension Spending (% of GDP)	b	6.8%

4.2 Results

Result 1: The effect depends on the probability of benchmark real rate reversion.

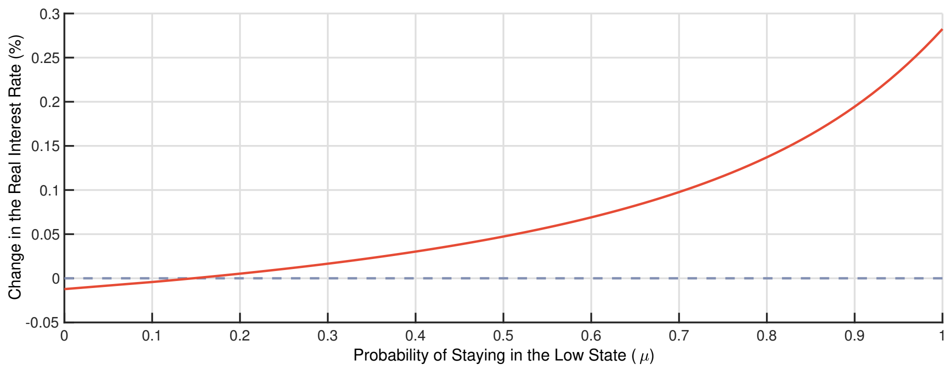
Figure 8 shows how the effect of a debt-financed fiscal expansion on the natural real interest rate depends on the probability of staying in the low state in the next period, μ. This computation holds the initial debt-to-GDP ratio fixed at 110 percent. As with the baseline model, when this probability is 1, higher debt unambiguously increases the natural real rate. At the other extreme, when this probability is 0, higher debt unambiguously has the perverse effect of reducing the natural real interest rate. The threshold μ^* is 0.14, which translates to an expected duration of 1.16 years.[13]

This means that if the economy is currently in the low state with a negative benchmark real interest rate and debt at 110 percent of GDP, which is close to the current level of debt in the United States, then a debt-financed fiscal expansion will be successful in raising the natural real interest rate only if agents expect the low state to last more than 1.16 years. If they expect otherwise, then the policy of increasing government debt will not have the desired effect of raising the real interest rate. In fact, it will worsen the problem by lowering

[13] Under assumption 1, the expected duration of the low state is $\frac{1}{1-\mu}$. See Appendix C.4 for a derivation.

the real rate further.[14]

Figure 8: Effect of increasing debt on the low-state natural real interest rate



Source: Author's calculations.
Notes: This figure plots the change in the real interest rate (in percent) against different probabilities of staying in the low state in the next period, ranging from 0 to 1. The computation holds the initial debt-to-GDP ratio fixed at 110 percent. Since this is a linearized model, the y-axis denotes the change in the linearized natural real interest rate in response to a change in the linearized government debt.

It is likely that since the aftermath of the Great Recession, when the economy experienced low (possibly negative) market-clearing real interest rates for a period of five to seven years, households in the United States expect the low state to last longer than 1.16 years. This suggests that there may still be room to use government debt to revive the natural real interest rate. However, due to the non-monotonicity of the effect illustrated in the baseline model, this stimulus is likely to work only up to a limit. We explore this limit next.

Result 2: The effect is non-monotonic with respect to the initial debt-to-GDP ratio.

Conditional on μ, the effect of higher public debt on the natural real interest rate depends on the initial debt-to-GDP ratio. Suppose we choose μ to be 0.5. Figure 9 shows the effect of increasing debt on the natural real interest rate for different initial debt-to-GDP ratios[15]

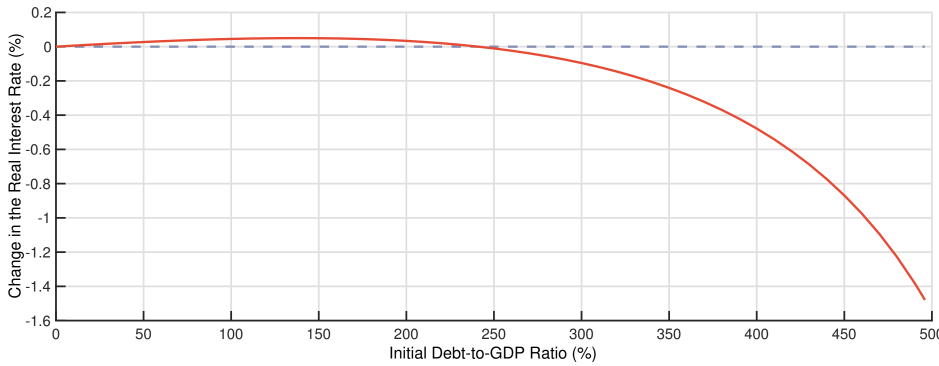
[14] For any given initial debt-to-GDP ratio, the threshold μ^* depends positively on the Frisch elasticity of labor supply. The logic follows exactly as discussed for the baseline model. A sensitivity analysis with respect to this elasticity is presented in Table 6 and Figure 14 in Appendix C.

[15] Since I am working with a linearized model, this corresponds to changing the steady state around which the system is approximated.

for the case where households expect the low state to last two years.[16]

We see that under the chosen μ, the effect of debt is positive and increasing up to the limit of 150 percent of GDP, beyond which it begins to decline while still remaining positive. As debt continues to increase, it reaches another limit of 250 percent of GDP, beyond which any further increases have a perverse effect on the economy. Once debt crosses this threshold, increasing debt lowers the natural real rate rather than raising it.

Figure 9: Non-monotonicity in the effect of debt



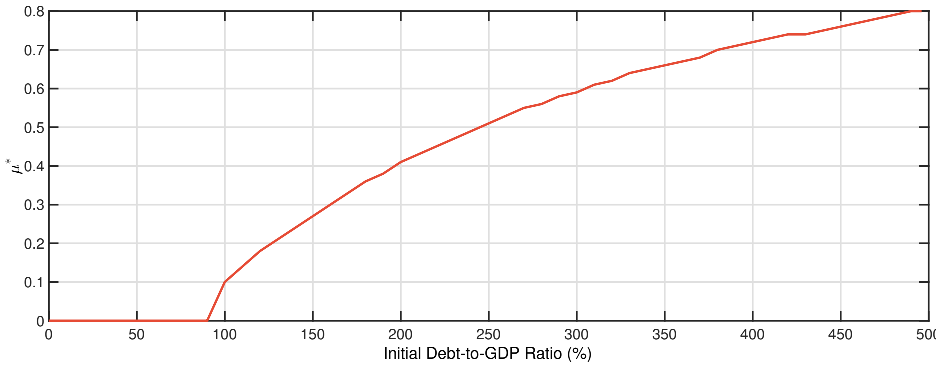
Source: Author's calculations.
Notes: This figure plots the change in the annualized real interest rate (in percent) against different initial debt-to-GDP ratios. The computation holds the probability of staying in the low state in the next period, μ, fixed at 0.5. Since this is a linearized model, the y-axis denotes the change in the linearized natural real interest rate in response to a change in the linearized government debt.

Finally, Figure 10 plots the threshold μ^* corresponding to different initial debt-to-GDP ratios. We see that for an economy with initial debt ranging from 0 percent to 90 percent of GDP, a debt-financed fiscal expansion always increases the real interest rate, regardless of the probability of benchmark real interest rate normalization (as long as it is not less than 0). After the initial debt-to-GDP ratio crosses this threshold of 90 percent, expectations about real interest rate normalization begin to have some bite in the model. Interestingly, 90 percent of GDP is also the critical value in Reinhart and Rogoff (2010) for debt to have a systemic negative effect on economic growth empirically.[17]

[16]The realized duration of the low state may be shorter or longer than two years. The expected duration simply indicates that in every period that the economy is in the low state, the model agents expect to be in that state for another two years.

[17]This paper analyzes public debt in a real model where debt has no impact on growth, but it is interesting

Figure 10: Variation in the threshold probability of reversion with respect to the initial debt-to-GDP ratio



Source: Author's calculations.
Notes: This figure plots the threshold probability of staying in the low state, μ^*, for different initial debt-to-GDP ratios. As μ^* increases, the perverse effect of debt is present for a larger range of μ values.

The intuition behind this result is most easily understood by flipping the graph 90 degrees, so that μ is on the x-axis and the initial debt-to-GDP ratio (which becomes the threshold debt-to-GDP ratio for the corresponding μ on the x-axis) is on the y-axis. The flipped graph reveals that as the probability of staying in the low state increases, the threshold debt-to-GDP ratio, up to which higher debt increases the real interest rate, correspondingly increases. This is because as μ increases, $(1 - \mu)$ decreases, which means that the weight attached to the negative expectations coming from the "bad" business-as-usual state is lower. Therefore, the threshold debt-to-GDP ratio for the overall effect of debt to be perverse must also be greater.[18]

5 Conclusion

This paper shows that in an environment with uncertainty about the path of future real interest rates emanating from unforeseeable forces, government debt may not be the silver bullet to solve the problem of low real interest rates, as it is made out to be. In analyzing

that the perverse effect of debt in their paper (in terms of economic growth) begins at a limit similar to the limit for the perverse effect of debt in my paper (in terms of the natural real interest rate).

[18]This is also shown in Table 5 in Appendix C.

the effectiveness of public debt, it is important to account for tradeoffs created by future contingencies where real interest rates are high and debt is burdensome. The argument is first formalized within a non-Ricardian framework with overlapping generations, and then quantitative implications are derived from a medium-scale version of the model.

Since this paper focuses on a low interest rate environment, it abstracts away from considerations of sovereign default, which are often cited as a reason for caution against buildups of large stocks of debt. Appendix D shows a simple way of adding sovereign default to the baseline model. The possibility of default imposes a risk premium on government debt, and an output cost if default is realized. This makes the bad business-as-usual state worse and exaggerates the potential perverse effect of government debt. This paper also abstracts away from nominal frictions to keep the discussion focused. Adding these frictions can yield important quantitative insights about the stimulative effects of debt policy conditional on the expected duration of the ZLB episode, which, in turn, can be inferred from survey data. Finally, it would also be interesting to empirically analyze how current fiscal policy affects expectations of agents regarding future fiscal policy, conditional on the existing level of government debt. This has been studied recently in the context of Japan (see Appendix E for a summary of the evidence), however, the literature in this area is scant. In a related study, Roth and Wohlfart (2019) examine how beliefs about the debt-to-GDP ratio affect people's attitudes toward government spending and taxation. They find that people underestimate the debt-to-GDP ratio and favor a cut in government spending once they learn about the actual amount of debt, but they do not alter their attitudes toward taxation.

The implications of this paper are relevant also for the current US situation and uncertainty created by the COVID-19 pandemic. The nominal interest is again very close to 0 percent, possibly with a negative natural real interest rate driving the most recent rate cut decision (March 2020) of the Federal Reserve Bank. There is definite uncertainty about the future path of the natural real interest rate absent fiscal policy intervention—no one knows how long the pandemic will last, or how great its effect on the economy will be. The government has taken on additional debt to push tax rebates to households. This paper suggests that the government should devise a strategy to manage household expectations in order to ensure the maximum benefit of this debt-financed fiscal expansion in stabilizing the natural real interest rate.

References

Alesina, Alberto, and Silvia Ardagna. 1998. "Tales of Fiscal Adjustment." *Economic Policy*, 13(27): 488–545.

Altonji, Joseph G. 1986. "Intertemporal Substitution in Labor Supply: Evidence from Micro Data." *Journal of Political Economy*, 94(3, Part 2): S176–S215.

Auerbach, Alan J, and Laurence J Kotlikoff. 1987. *Dynamic Fiscal Policy.* Cambridge University Press.

Chetty, Raj, Adam Guren, Day Manoli, and Andrea Weber. 2011. "Are Micro and Macro Labor Supply Elasticities Consistent? A Review of Evidence on the Intensive and Extensive Margins." *American Economic Review*, 101(3): 471–75.

Christiano, Lawrence, Martin Eichenbaum, and Sergio Rebelo. 2011. "When Is the Government Spending Multiplier Large?" *Journal of Political Economy*, 119(1): 78–121.

Coibion, Olivier, Yuriy Gorodnichenko, and Johannes Wieland. 2012. "The Optimal Inflation Rate in New Keynesian Models: Should Central Banks Raise Their Inflation Targets in Light of the Zero Lower Bound?" *Review of Economic Studies*, 79(4): 1371–1406.

Denes, Matthew, Gauti B Eggertsson, and Sophia Gilbukh. 2013. "Deficits, Public Debt Dynamics and Tax and Spending Multipliers." *The Economic Journal*, 123(566): F133–F163.

D'Erasmo, Pablo, and Enrique G Mendoza. 2016. "Optimal Domestic (and External) Sovereign Default." National Bureau of Economic Research Working Paper No. 22509.

Eggertsson, Gauti B. 2011. "What Fiscal Policy Is Effective at Zero Interest Rates?" *NBER Macroeconomics Annual*, 25(1): 59–112.

Eggertsson, Gauti B, and Neil R Mehrotra. 2014. "A Model of Secular Stagnation." National Bureau of Economic Research Working Paper No. 20574.

Eggertsson, Gauti B, and Vaishali Garga. 2019. "Sticky Prices versus Sticky Information: Does it Matter for Policy Paradoxes?" *Review of Economic Dynamics*, 31: 363–392.

Eggertsson, Gauti B, Neil R Mehrotra, and Jacob A Robbins. 2019. "A Model of Secular Stagnation: Theory and Quantitative Evaluation." *American Economic Journal: Macroeconomics*, 11(1): 1–48.

Eggertsson, Gauti B, Sergey Egiev, Alessandro Lin, Josef Platzer, and Luca Riva. 2020. "A Toolkit for Solving Models with a Lower Bound on Interest Rates of Stochastic Duration." *SSRN Working Paper No. 3532643.*

Erceg, Christopher, and Jesper Lindé. 2014. "Is there a Fiscal Free Lunch in a Liquidity Trap?" *Journal of the European Economic Association*, 12(1): 73–107.

Fatás, Antonio, and Lawrence H Summers. 2018. "The Permanent Effects of Fiscal Consolidations." *Journal of International Economics*, 112: 238–250.

Guajardo, Jaime, Daniel Leigh, and Andrea Pescatori. 2014. "Expansionary Austerity? International Evidence." *Journal of the European Economic Association*, 12(4): 949–968.

Hall, Robert E. 2009. "Reconciling Cyclical Movements in the Marginal Value of Time and the Marginal Product of Labor." *Journal of Political Economy*, 117(2): 281–323.

Heer, Burkhard, and Alfred Maussner. 2009. *Dynamic General Equilibrium Modeling: Computational Methods and Applications.* Springer Science & Business Media.

Klein, Paul. 2000. "Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model." *Journal of Economic Dynamics and Control*, 24(10): 1405–1423.

Kobayashi, Keiichiro, and Kozo Ueda. 2017. "Secular Stagnation under the Fear of a Government Debt Disaster." *The Canon Institute for Global Studies Working Paper.*

Kocherlakota, Narayana. 2015. "Public Debt and the Long-run Neutral Real Interest Rate." *Bank of Korea Conference Remarks.*

MaCurdy, Thomas E. 1981. "An Empirical Model of Labor Supply in a Life-cycle Setting." *Journal of Political Economy*, 89(6): 1059–1085.

Masayuki, Morikawa. 2017. "Impact of Policy Uncertainty on Consumption and Saving Behavior: Evidence from a Survey on Consumers." *RIETI Discussion Paper No. 17075.*

Mehrotra, Neil R. 2017. "Debt Sustainability in a Low Interest Rate World." *Hutchins Center Working Paper No. 32.*

Nickel, Christiane, and Andreas Tudyka. 2013. "Fiscal Stimulus in Times of High Public Debt: Reconsidering Multipliers and Twin Deficits." *VOX, CEPRs Policy Portal.*

Perotti, Roberto. 1999. "Fiscal Policy in Good Times and Bad." *The Quarterly Journal of Economics*, 114(4): 1399–1436.

Reichling, Felix, and Charles Whalen. 2012. "Review of Estimates of the Frisch Elasticity of Labor Supply." *Congressional Budget Office Working Paper.*

Reinhart, Carmen M, and Kenneth S Rogoff. 2010. "Growth in a Time of Debt." *American Economic Review*, 100(2): 573–78.

Rogerson, Richard, and Johanna Wallenius. 2009. "Micro and Macro Elasticities in a Life Cycle Model with Taxes." *Journal of Economic Theory*, 144(6): 2277–2292.

Roth, Christopher, and Johannes Wohlfart. 2019. "How Do Expectations about the Macroeconomy Affect Personal Expectations and Behavior?" *Review of Economics and Statistics*, 1–45.

Smets, Frank, and Rafael Wouters. 2007. "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach." *American Economic Review*, 97(3): 586–606.

Summers, Lawrence H. 2015. "Demand Side Secular Stagnation." *American Economic Review*, 105(5): 60–65.

Woodford, Michael. 2011. "Simple Analytics of the Government Expenditure Multiplier." *American Economic Journal: Macroeconomics*, 3(1): 1–35.

Woodford, Michael, and Gauti Eggertsson. 2003. "The Zero Bound on Interest Rates and Optimal Monetary Policy." *Brookings Papers on Economic Activity*, 1: 139–233.

A Appendix: Illustrative Model

A.1 Model Equilibrium

First-order conditions of the household optimization problem:

$$(c_t^m)^{-\gamma} + \lambda_t = 0$$
$$\beta D_t \mathbb{E}_t (c_{t+1}^o)^{-\gamma} + \lambda_{t+1} = 0$$
$$\lambda_t - \lambda_{t+1}(1 + r_t) = 0$$

Combining the above three yields the Euler equation:

$$(c_t^m)^{-\gamma} = \beta D_t (1 + r_t) \mathbb{E}_t (c_{t+1}^o)^{-\gamma}$$

First-order conditions of the firm optimization problem:

$$w_t = (1 - \alpha)\frac{Y_t}{L_t}$$
$$Z_t = Y_t - w_t L_t$$
$$L_t = \bar{L}$$

Government:

$$T_t^m = \theta_\tau \left[(1 + r_{t-1})B_{t-1}^g + G_t + B_t^g\right]$$
$$T_t^o = (1 - \theta_\tau) \left[(1 + r_{t-1})B_{t-1}^g + G_t + B_t^g\right]$$

Equilibrium:
The flexible-price competitive equilibrium in the economy is defined as a sequence of endogenous variables $\{Y_t, c_t^m, c_t^o, L_t, w_t, T_t^m, T_t^o, Z_t, r_t, a_t^m\}$, which satisfies the following equations, given exogenous $\{B_t^g, G_t, D_t\}$ and labor endowment $\{\bar{L}\}$.

$$(c_t^m)^{-\gamma} = \beta D_t (1 + r_t) \mathbb{E}_t (c_{t+1}^o)^{-\gamma}$$
$$c_t^m = w_t \theta L_t + \theta_z Z_t - T_t^m - a_t^m$$
$$c_{t+1}^o = w_{t+1}(1 - \theta)\bar{L} + (1 - \theta_z) Z_{t+1} - T_{t+1}^o + (1 + r_t) a_t^m$$
$$Y_t = {L_t}^{1-\alpha}$$
$$L_t = \bar{L}$$
$$w_t = (1 - \alpha) L_t^{-\alpha}$$
$$Z_t = Y_t - w_t L_t$$
$$T_t^m = \theta_\tau \left[(1 + r_{t-1})B_{t-1}^g + G_t - B_t^g\right]$$
$$T_t^o = (1 - \theta_\tau) \left[(1 + r_{t-1})B_{t-1}^g + G_t - B_t^g\right]$$
$$a_t^m = B_t^g$$

Under assumptions 2 and 3, the equilibrium system reduces to:

$$(c_t^m)^{-\gamma} = \beta D_t(1+r_t)\mathbb{E}_t(c_{t+1}^o)^{-\gamma}$$
$$c_t^m = w_t\theta\bar{L} + \theta_z Z_t - T_t^m - a_t^m$$
$$c_{t+1}^o = w_{t+1}(1-\theta)\bar{L} + (1-\theta_z)Z_{t+1} - T_{t+1}^o + (1+r_t)a_t^m$$
$$Y_t = \bar{L}^{1-\alpha} - \nu_g\mathbb{W}_t, \quad s.t. \quad \nu_g = 0 \text{ if low state}, \nu_g > 0 \text{ if bau state}$$
$$w_t = (1-\alpha)\bar{L}^{-\alpha}$$
$$Z_t = Y_t - w_t L_t$$
$$T_t^m = \theta_\tau\left[\mathbb{W}_{t-1} - \frac{\mathbb{W}_t}{1+r_t} + G_t\right]$$
$$T_t^o = (1-\theta_\tau)\left[\mathbb{W}_{t-1} - \frac{\mathbb{W}_t}{1+r_t} + G_t\right]$$
$$a_t^m = \frac{\mathbb{W}_t}{1+r_t}$$

Note that: $w_t = w(\bar{L}, \nu_g, \mathbb{W}_t); Z_t = Z(\bar{L}, \nu_g, \mathbb{W}_t); T_t^m = Tm(\theta_\tau, \mathbb{W}_{t-1}, \mathbb{W}_t, r_t, G_t); T_t^o = To((\theta_\tau, \mathbb{W}_{t-1}, \mathbb{W}_t, r_t, G_t)$. From the middle-aged budget constraint: $c_t^m = cm(w_t, \bar{L}, \theta_z, Z_t, T_t^m, a_t^m) = f(\bar{L}, \nu_g, \mathbb{W}_t, \theta_\tau, \mathbb{W}_{t-1}, r_t, G_t, a_t^m)$. Similarly, from the old-age budget constraint: $c_t^o = g(\bar{L}, \nu_g, \mathbb{W}_t, \theta_\tau, \mathbb{W}_{t-1}, r_t, G_t, a_t^m)$. From the asset market-clearing condition, we know that $a_t^m = \frac{\mathbb{W}_t}{1+r_t}$.

Define the parameter set $\Theta \equiv \{\beta, \gamma, \alpha, \bar{L}, \theta_z, \theta_\tau\}$. This implies:

$$c_t^m = f(\nu_g, \mathbb{W}_t, \mathbb{W}_{t-1}, r_t, G_t; \Theta)$$

$$c_t^o = f(\nu_g, \mathbb{W}_t, \mathbb{W}_{t-1}, r_t, G_t; \Theta)$$

Since $G_t = \bar{G} \equiv 0$, this further reduces to:

$$c_t^m = f(\nu_g, \mathbb{W}_t, \mathbb{W}_{t-1}, r_t; \Theta)$$

$$c_t^o = f(\nu_g, \mathbb{W}_t, \mathbb{W}_{t-1}, r_t; \Theta)$$

Thus the entire system can be reduced to one equation, that is, the Euler equation in terms of one endogenous variable $\{r_t\}$, given exogenous $\{\mathbb{W}_t, D_t\}$ and parameter set $\{\Theta, \nu_g, \bar{G}\}$:

$$\left\{(1-\alpha)\theta\bar{L}^{1-\alpha} + \theta_z\left(\bar{L}^{1-\alpha} - \nu_g\mathbb{W}_t - (1-\alpha)\bar{L}^\alpha\right) - \theta_\tau\left[\mathbb{W}_{t-1} - \frac{\mathbb{W}_t}{1+r_t}\right] - \frac{\mathbb{W}_t}{1+r_t}\right\}^{-\gamma}$$
$$= \beta D_t(1+r_t)\mathbb{E}_t\left\{(1-\alpha)(1-\theta)\bar{L}^{1-\alpha} + (1-\theta_z)\left(\bar{L}^{1-\alpha} - \nu_g\mathbb{W}_{t+1} - (1-\alpha)\bar{L}^\alpha\right) - (1-\theta_\tau)\left[\mathbb{W}_t - \frac{\mathbb{W}_{t+1}}{1+r_{t+1}}\right] + \mathbb{W}_t\right\}^{-\gamma}$$

Here, $\nu_g = 0$ in the low state while $\nu_g > 0$ in the business-as-usual state.

A.2 Proof of Proposition 1

Proof. Under assumption 1, once the economy reverts to the business-as-usual state, it stays there forever. Therefore, in the constant solution of the business-as-usual state with $D_t = D_B$, the Euler equation becomes:

$$\left\{(1-\alpha)\theta\bar{L}^{1-\alpha} + \theta_z\left(\bar{L}^{1-\alpha} - \nu_g\mathbb{W}^* - (1-\alpha)\bar{L}^\alpha\right) - \theta_\tau\left[\mathbb{W}^* - \frac{\mathbb{W}^*}{1+r_B}\right] - \frac{\mathbb{W}^*}{1+r_B}\right\}^{-\gamma}$$
$$= \beta D_B(1+r_B)\left\{(1-\alpha)(1-\theta)\bar{L}^{1-\alpha} + (1-\theta_z)\left(\bar{L}^{1-\alpha} - \nu_g\mathbb{W}^* - (1-\alpha)\bar{L}^\alpha\right) - (1-\theta_\tau)\left[\mathbb{W}^* - \frac{\mathbb{W}^*}{1+r_B}\right] + \mathbb{W}^*\right\}^{-\gamma}$$

This implies:

$$r_B = f\left(\mathbb{W}^*, \nu_g, D_B; \Theta\right)$$

Under assumption 1, in every period of the low state, there is a probability μ of being in the low state in the next period while there is a probability $(1-\mu)$ of the reverting to the business-as-usual state in the next period. Therefore, in the constant solution of the low state with $D_t = D_L$ and $\nu_g = 0$, the Euler equation is:

$$(c_L^m)^{-\gamma} = \beta D_L(1+r_L)\left[\mu(c_L^o)^{-\gamma} + (1-\mu)(c_B^o)^{-\gamma}\right]$$

This implies:

$$\begin{aligned}
&\left\{(1-\alpha)\theta\bar{L}^{1-\alpha} + \theta_z\left(\bar{L}^{1-\alpha} - (1-\alpha)\bar{L}^{\alpha}\right) - \theta_\tau\left[\mathbb{W}^* - \frac{\mathbb{W}^*}{1+r_L}\right] - \frac{\mathbb{W}^*}{1+r_L}\right\}^{-\gamma} \\
&= \beta D_L(1+r_L)\Bigg[\mu\left\{(1-\alpha)(1-\theta)\bar{L}^{1-\alpha} + (1-\theta_z)\left(\bar{L}^{1-\alpha} - (1-\alpha)\bar{L}^{\alpha}\right) - (1-\theta_\tau)\left[\mathbb{W}^* - \frac{\mathbb{W}^*}{1+r_L}\right] + \mathbb{W}^*\right\}^{-\gamma} \\
&+ (1-\mu)\left\{(1-\alpha)(1-\theta)\bar{L}^{1-\alpha} + (1-\theta_z)\left(\bar{L}^{1-\alpha} - \nu_g\mathbb{W}^* - (1-\alpha)\bar{L}^{\alpha}\right) - (1-\theta_\tau)\left[\mathbb{W}^* - \frac{\mathbb{W}^*}{1+r_B}\right] + \mathbb{W}^*\right\}^{-\gamma}\Bigg]
\end{aligned}$$

This implies:

$$r_L = g\left(\mathbb{W}^*, D_L, \mu, \nu_g, r_B; \Theta\right) = g\left(\mathbb{W}^*, D_L, \mu, \nu_g, D_B; \Theta\right)$$

□

A.3 Proof of Proposition 2

Proof. Differentiating the above Euler equation, of the low-state constant solution, with respect to $\mathbb{W}^*$ yields:

$$\begin{aligned}
&-\gamma\left(c_L^m\right)^{-\gamma-1}\left[-\theta_\tau\left(1 - \frac{1}{1+r_L}\right) - \frac{1}{1+r_L} - \frac{\mathbb{W}^*(\theta_\tau - 1)}{(1+r_L)^2}\frac{dr_L}{d\mathbb{W}^*}\right] \\
&= \beta D_L\left[\mu(c_L^o)^{-\gamma} + (1-\mu)(c_B^o)^{-\gamma}\right]\frac{dr_L}{d\mathbb{W}^*} \\
&+ \beta D_L(1+r_L)\mu(-\gamma)\left(c_L^o\right)^{-\gamma-1}\left[-(1-\theta_\tau)\left(1 - \frac{1}{1+r_L}\right) + 1 - \frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_L)^2}\frac{dr_L}{d\mathbb{W}^*}\right] \\
&+ \beta D_L(1+r_L)(1-\mu)(-\gamma)\left(c_B^o\right)^{-\gamma-1}\left[(1-\theta_z)(-\nu_g) - (1-\theta_\tau)\left(1 - \frac{1}{1+r_B}\right) + 1 - \frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}\right]
\end{aligned}$$

This implies:

$$\frac{dr_L}{d\mathbb{W}^*} = \frac{num_L}{den_L},$$

where,

$$\begin{aligned}
num_L &= \beta D_L(1+r_L)\mu(-\gamma)\left(c_L^o\right)^{-\gamma-1}\left[-(1-\theta_\tau)\left(1 - \frac{1}{1+r_L}\right) + 1\right] \\
&\quad + \beta D_L(1+r_L)(1-\mu)(-\gamma)\left(c_B^o\right)^{-\gamma-1}\left[(1-\theta_z)(-\nu_g) - (1-\theta_\tau)\left(1 - \frac{1}{1+r_B}\right) + 1 - \frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}\right] \\
&\quad + \gamma\left(c_L^m\right)^{-\gamma-1}\left[-\theta_\tau\left(1 - \frac{1}{1+r_L}\right) - \frac{1}{1+r_L}\right], \\
den_L &= \gamma\left(c_L^m\right)^{-\gamma-1}\frac{\mathbb{W}^*(\theta_\tau - 1)}{(1+r_L)^2} - \beta D_L\left[\mu(c_L^o)^{-\gamma} + (1-\mu)(c_B^o)^{-\gamma}\right] + \beta D_L(1+r_L)\mu(-\gamma)\left(c_L^o\right)^{-\gamma-1}\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_L)^2}
\end{aligned}$$

The total implicit effect can be decomposed into two parts—the direct effect and the indirect effect—by combining the terms multiplied with μ versus $(1-\mu)$ in the total derivative of the Euler equation wrt $\mathbb{W}^*$

as follows:

$$
\begin{aligned}
&-\gamma\left(c_L^m\right)^{-\gamma-1}\left[-\theta_\tau\left(1-\frac{1}{1+r_L}\right)-\frac{1}{1+r_L}-\frac{\mathbb{W}^*(\theta_\tau-1)}{(1+r_L)^2}\frac{dr_L}{d\mathbb{W}^*}\right] \\
&=\mu\left\{\beta D_L(c_L^o)^{-\gamma}\frac{dr_L}{d\mathbb{W}^*}+\beta D_L(1+r_L)(-\gamma)\left(c_L^o\right)^{-\gamma-1}\left[-(1-\theta_\tau)\left(1-\frac{1}{1+r_L}\right)+1-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_L)^2}\frac{dr_L}{d\mathbb{W}^*}\right]\right\}+ \\
&(1-\mu)\left\{\beta D_L(c_B^o)^{-\gamma}\frac{dr_L}{d\mathbb{W}^*}+\beta D_L(1+r_L)(-\gamma)\left(c_B^o\right)^{-\gamma-1}\left[(1-\theta_z)(-\nu_g)-(1-\theta_\tau)\left(1-\frac{1}{1+r_B}\right)+1-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}\right]\right\}
\end{aligned}
$$

Thus, we have:

$$
\left(\frac{dr_L}{d\mathbb{W}^*}\right)^{direct}=\frac{\beta D_L(1+r_L)(-\gamma)\left(c_L^o\right)^{-\gamma-1}\left[-(1-\theta_\tau)\left(1-\frac{1}{1+r_L}\right)+1\right]+\gamma\left(c_L^m\right)^{-\gamma-1}\left[-\theta_\tau\left(1-\frac{1}{1+r_L}\right)-\frac{1}{1+r_L}\right]}{\gamma\left(c_L^m\right)^{-\gamma-1}\frac{\mathbb{W}^*(\theta_\tau-1)}{(1+r_L)^2}-\mu\beta D_L(c_L^o)^{-\gamma}+\mu\beta D_L(1+r_L)(-\gamma)\left(c_L^o\right)^{-\gamma-1}\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_L)^2}-(1-\mu)\beta D_L(c_B^o)^{-\gamma}},
$$

$$
\left(\frac{dr_L}{d\mathbb{W}^*}\right)^{indirect}=\frac{\beta D_L(1+r_L)(-\gamma)\left(c_B^o\right)^{-\gamma-1}\left[(1-\theta_z)(-\nu_g)-(1-\theta_\tau)\left(1-\frac{1}{1+r_B}\right)+1-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}\right]+\gamma\left(c_L^m\right)^{-\gamma-1}\left[-\theta_\tau\left(1-\frac{1}{1+r_L}\right)-\frac{1}{1+r_L}\right]}{\gamma\left(c_L^m\right)^{-\gamma-1}\frac{\mathbb{W}^*(\theta_\tau-1)}{(1+r_L)^2}-\mu\beta D_L(c_L^o)^{-\gamma}+\mu\beta D_L(1+r_L)(-\gamma)\left(c_L^o\right)^{-\gamma-1}\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_L)^2}-(1-\mu)\beta D_L(c_B^o)^{-\gamma}}
$$

At this stage, we can verify by substitution that:

$$
\frac{dr_L}{d\mathbb{W}^*}=\mu\underbrace{\left(\frac{dr_L}{d\mathbb{W}^*}\right)^{direct}}_{=d(D_L,\mathbb{W}^*,\nu_g,D_B;\Theta)}+(1-\mu)\underbrace{\left(\frac{dr_L}{d\mathbb{W}^*}\right)^{indirect}}_{=i\left(D_L,\mathbb{W}^*,\nu_g,D_B,\frac{dr_B}{d\mathbb{W}^*};\Theta\right)}
$$

If $-1<r_L<0$ and given that parameters $\theta,\theta_z,\theta_\tau,\alpha\in[0,1]$, this implies that the denominator (of the direct and indirect effect) is negative.
For the direct effect, the numerator can be rearranged to:

$$
\beta D_L(1+r_L)\gamma\left(c_L^o\right)^{-\gamma-1}\left[(1-\theta_\tau)\left(\frac{r_L}{1+r_L}\right)-1\right]-\gamma\left(c_L^m\right)^{-\gamma-1}\theta_\tau-\gamma\left(c_L^m\right)^{-\gamma-1}\frac{1-\theta_\tau}{1+r_L}<0
$$

This implies: $\left(\frac{dr_L}{d\mathbb{W}^*}\right)^{direct}\geq 0$.

For the indirect effect, the numerator can be rearranged to:

$$
\begin{aligned}
&\beta D_L(1+r_L)\gamma\left(c_B^o\right)^{-\gamma-1}\left[(1-\theta_z)\nu_g+(1-\theta_\tau)\left(\frac{r_B}{1+r_B}\right)+\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}\right] \\
&-\beta D_L(1+r_L)\gamma\left(c_B^o\right)^{-\gamma-1}-\gamma\left(c_L^m\right)^{-\gamma-1}\theta_\tau-\gamma\left(c_L^m\right)^{-\gamma-1}\left(\frac{1-\theta_\tau}{1+r_L}\right)
\end{aligned}
$$

A large enough ν_g implies the numerator>0, which implies $\left(\frac{dr_L}{d\mathbb{W}^*}\right)^{indirect}\leq 0$. In particular, we need:

$$
\begin{aligned}
\nu_g &> \frac{\frac{\beta D_L(1+r_L)\gamma(c_B^o)^{-\gamma-1}+\gamma(c_L^m)^{-\gamma-1}\left[\theta_\tau+\left(\frac{1-\theta_\tau}{1+r_L}\right)\right]}{\beta D_L(1+r_L)\gamma\left(c_B^o\right)^{-\gamma-1}}-(1-\theta_\tau)\left(\frac{r_B}{1+r_B}\right)-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}}{1-\theta_z} \\
&= \frac{\frac{(c_L^m)^{-\gamma-1}\left[\theta_\tau+\left(\frac{1-\theta_\tau}{1+r_L}\right)\right]}{\beta D_L(1+r_L)\left(c_B^o\right)^{-\gamma-1}}-(1-\theta_\tau)\left(\frac{r_B}{1+r_B}\right)-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}+1}{1-\theta_z}
\end{aligned}
$$

A.4 Proof of Proposition 3

To begin with, note that $den_L = denominator\left[\left(\frac{dr_L}{d\mathbb{W}^*}\right)^{direct}\right] = denominator\left[\left(\frac{dr_L}{d\mathbb{W}^*}\right)^{indirect}\right] < 0$. This means that:

$$num_L > 0 \implies \frac{dr_L}{d\mathbb{W}^*} = \frac{num_L}{den_L} < 0$$

$$\begin{aligned}
num_L = &\beta D_L(1+r_L)\mu(-\gamma)\left(c_L^o\right)^{-\gamma-1}\left[-(1-\theta_\tau)\left(1-\frac{1}{1+r_L}\right)+1\right] \\
&+\beta D_L(1+r_L)(1-\mu)(-\gamma)\left(c_B^o\right)^{-\gamma-1}\left[(1-\theta_z)(-\nu_g)-(1-\theta_\tau)\left(1-\frac{1}{1+r_B}\right)+1-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}\right] \\
&+\gamma\left(c_L^m\right)^{-\gamma-1}\left[-\theta_\tau\left(1-\frac{1}{1+r_L}\right)-\frac{1}{1+r_L}\right] > 0
\end{aligned}$$

$$\begin{aligned}
\equiv \mu\Bigg\{&\beta D_L(1+r_L)(-\gamma)\left(c_L^o\right)^{-\gamma-1}\left[-(1-\theta_\tau)\left(1-\frac{1}{1+r_L}\right)+1\right] \\
&-\beta D_L(1+r_L)(-\gamma)\left(c_B^o\right)^{-\gamma-1}\left[(1-\theta_z)(-\nu_g)-(1-\theta_\tau)\left(1-\frac{1}{1+r_B}\right)+1-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}\right]\Bigg\} \\
&>\gamma\left(c_L^m\right)^{-\gamma-1}\left[\theta_\tau\left(1-\frac{1}{1+r_L}\right)+\frac{1}{1+r_L}\right] \\
&+\beta D_L(1+r_L)\gamma\left(c_B^o\right)^{-\gamma-1}\left[(1-\theta_z)(-\nu_g)-(1-\theta_\tau)\left(1-\frac{1}{1+r_B}\right)+1-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}\right]
\end{aligned}$$

Thus, we can define μ^* as:

$$\begin{aligned}
\mu^* &= \frac{\gamma\left(c_L^m\right)^{-\gamma-1}\left[\theta_\tau\left(1-\frac{1}{1+r_L}\right)+\frac{1}{1+r_L}\right]+\beta D_L(1+r_L)\gamma\left(c_B^o\right)^{-\gamma-1}\left[(1-\theta_z)(-\nu_g)-(1-\theta_\tau)\left(1-\frac{1}{1+r_B}\right)+1-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}\right]}{\beta D_L(1+r_L)(-\gamma)\left(c_L^o\right)^{-\gamma-1}\left[-(1-\theta_\tau)\left(1-\frac{1}{1+r_L}\right)+1\right]-\beta D_L(1+r_L)(-\gamma)\left(c_B^o\right)^{-\gamma-1}\left[(1-\theta_z)(-\nu_g)-(1-\theta_\tau)\left(1-\frac{1}{1+r_B}\right)+1-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}\right]} \\
&= \frac{\gamma\left(c_L^m\right)^{-\gamma-1}\left(\theta_\tau+\frac{1-\theta_\tau}{1+r_L}\right)+\beta D_L(1+r_L)\gamma\left(c_B^o\right)^{-\gamma-1}\left[-\nu_g(1-\theta_z)-(1-\theta_\tau)\left(\frac{r_B}{1+r_B}\right)-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}+1\right]}{\beta D_L(1+r_L)\gamma\left\{\left(c_B^o\right)^{-\gamma-1}\left[-\nu_g(1-\theta_z)-(1-\theta_\tau)\left(\frac{r_B}{1+r_B}\right)-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}+1\right]+\left(c_L^o\right)^{-\gamma-1}\left[(1-\theta_\tau)\left(\frac{r_L}{1+r_L}\right)-1\right]\right\}} \\
&= \mu^*\left(r_L, D_L, \mathbb{W}^*, r_B, \frac{dr_B}{d\mathbb{W}^*}, \nu_g; \Theta\right)
\end{aligned}$$

such that if $denominator(\mu^*) < 0$, then $\mu < \mu^* \implies \frac{dr_L}{d\mathbb{W}^*} < 0$, while if $denominator(\mu^*) > 0$, then $\mu > \mu^* \implies \frac{dr_L}{d\mathbb{W}^*} < 0$. So what we need now is a condition such that $denominator(\mu^*) > 0$. In particular, we need:

$$\nu_g > \frac{\frac{\left(c_L^o\right)^{-\gamma-1}\left[(1-\theta_\tau)\left(\frac{r_L}{1+r_L}\right)-1\right]}{\left(c_B^o\right)^{-\gamma-1}}-(1-\theta_\tau)\left(\frac{r_B}{1+r_B}\right)-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}+1}{1-\theta_z}$$

Thus, the final condition we have is:

$$\nu_g > \frac{\max\left\{\frac{\left(c_L^m\right)^{-\gamma-1}\left[\theta_\tau+\left(\frac{1-\theta_\tau}{1+r_L}\right)\right]}{\beta D_L(1+r_L)\left(c_B^o\right)^{-\gamma-1}}, \frac{\left(c_L^o\right)^{-\gamma-1}\left[(1-\theta_\tau)\left(\frac{r_L}{1+r_L}\right)-1\right]}{\left(c_B^o\right)^{-\gamma-1}}\right\}-(1-\theta_\tau)\left(\frac{r_B}{1+r_B}\right)-\frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*}+1}{1-\theta_z}$$

Sanity check: Note that $numerator(\mu^*) = -numerator\left[\left(\frac{dr_L}{d\mathbb{W}^*}\right)^{indirect}\right]$. Since ν_g was chosen to be large enough so that the $numerator\left[\left(\frac{dr_L}{d\mathbb{W}^*}\right)^{indirect}\right] > 0$, this implies $numerator(\mu^*) < 0$. This verifies that $\mu^* > 0$, which is a necessary condition for this probability parameter to have any significance in the model.

A.5 Proof of Corollary 1

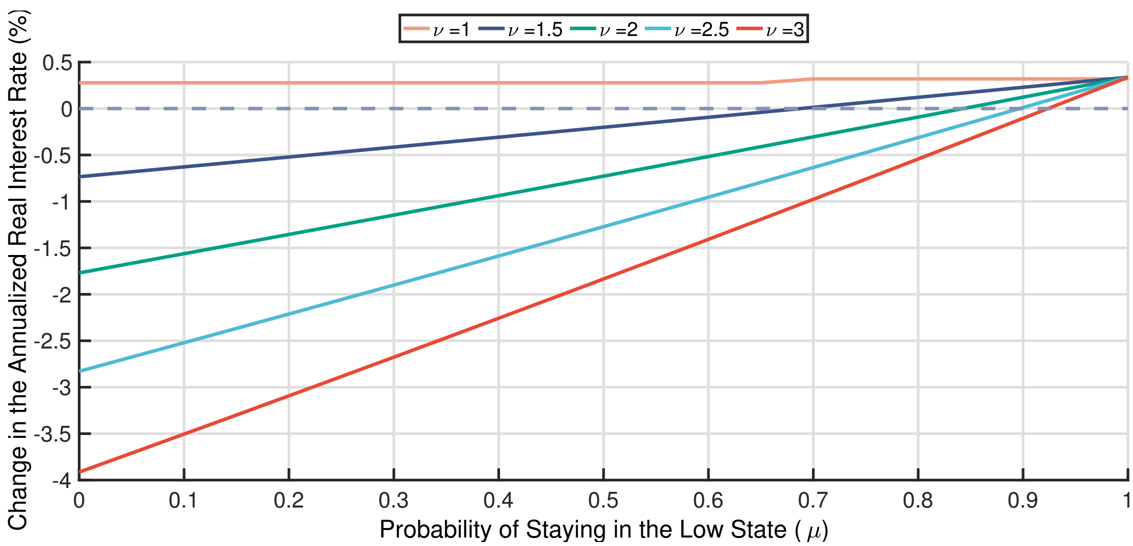
Proof. This follows directly from the solution for μ^*:

$$\mu^* = \frac{\gamma\left(c_L^m\right)^{-\gamma-1}\left(\theta_\tau + \frac{1-\theta_\tau}{1+r_L}\right) + \beta D_L(1+r_L)\gamma\left(c_B^o\right)^{-\gamma-1}\left[-\nu_g(1-\theta_z) - (1-\theta_\tau)\left(\frac{r_B}{1+r_B}\right) - \frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*} + 1\right]}{\beta D_L(1+r_L)\gamma\left\{\left(c_B^o\right)^{-\gamma-1}\left[-\nu_g(1-\theta_z) - (1-\theta_\tau)\left(\frac{r_B}{1+r_B}\right) - \frac{\mathbb{W}^*(1-\theta_\tau)}{(1+r_B)^2}\frac{dr_B}{d\mathbb{W}^*} + 1\right] + \left(c_L^o\right)^{-\gamma-1}\left[(1-\theta_\tau)\left(\frac{r_L}{1+r_L}\right) - 1\right]\right\}}$$

As ν_g increases, *numerator*(μ^*) and *denominator*(μ^*) both become equally more negative. This leads to an increase in μ^* (think for example, $\frac{-1}{-2} < \frac{-2}{-3}$). ☐

Figure 11 shows a graphical representation of this corollary under the illustrative parameterization described in the main text. It shows that for smaller values of ν_g, increasing debt has a positive effect on the low-state natural real interest rate, irrespective of μ. However, as ν_g increases, there emerges a threshold μ^* such that for $\mu < \mu^*$, increasing debt has a perverse effect on the natural real interest rate. This μ^* is a monotonically increasing function of ν_g.

Figure 11: Sensitivity analysis with respect to ν_g: Effect of increasing debt on the low-state natural real interest rate



Source: Author's calculations.
Notes: This figure plots the change in the annualized real interest rate (in percent) against different probabilities of staying in the low state in the next period, for a range of ν_g. The computation holds the initial annual debt-to-GDP ratio fixed at 3 percent.

A.6 Calibration

Table 4: Illustrative Parameters

Parameter	Description	Value
Illustrative Model		
D_B	Business-as-usual-state Inter-temporal Wedge Shock	1
r_B^b	Business-as-usual-state Benchmark Real Interest Rate	10% p.a.
r_L^b	Low-state Benchmark Real Interest Rate	-2% p.a.
$\bar{L}$	Labor Endowment	1
θ	Middle-aged Share of Labor Endowment	0.5
$(1-\alpha)$	Labor Share	0.7
θ_z	Middle-aged Share of Profits	0.5
θ_τ	Middle-aged Share of Taxes	0.5
γ	Coefficient of Relative Risk Aversion	1
ν_g	Output Cost of Debt in the Business-as-usual State	1.3
$\bar{G}$	Government Purchases	0
Endogenous Labor and Capital		
η	Frisch Elasticity of Labor	1.4
δ	Capital Depreciation Rate	10% p.a.
Sovereign Default Risk		
d_0	Risk Premium Parameter	0.05
d_1	Risk Premium Parameter	1
d_g	Output Cost of Default	0

B Appendix: Model with Endogenous Labor and Capital

Households:

With the addition of capital, households can now save in government bonds as well as capital, which is rented out to the firms in the same period. I assume full depreciation. This is a reasonable assumption, as every period in the model corresponds to 30 years in the real world. Capital is unlikely to serve as a store of wealth for such a long period of time.

$$U_t(c_t^m, c_{t+1}^o, h_t^m, h_{t+1}^o) = \max_{c_t^m, c_{t+1}^o} \left\{ \frac{1}{1-\gamma}(c_t^m)^{1-\gamma} + \beta D \mathbb{E}_t \frac{1}{1-\gamma}(c_{t+1}^o)^{1-\gamma} - \chi \frac{(h_t^m)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} - \chi \beta D_t \frac{(h_{t+1}^o)^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\}$$
$$\text{s.t.} \quad c_t^m = w_t h_t^m (1-\tau_t) - a_t^m - K_t^s + r_t^k (1-\tau_t) K_t^s$$
$$c_{t+1}^o = w_{t+1} h_{t+1}^o (1-\tau_{t+1}) + (1+r_t) a_t^m + (1-\delta) K_t^s \tag{1}$$

where h_t^m and h_{t+1}^o is labor supply of the middle-aged and old, respectively; η is the Frisch elasticity of labor; τ_t is the income tax rate; a_t^m is the supply of loanable funds that is invested in government bonds; and K_t^s is the supply of loanable funds that is invested in capital. I assume that when households are middle aged, they convert consumption goods into productive capital at a fixed 1:1 rate. They rent the capital to firms in the same period and receive rental income. In the next period of life, when they are old, they convert the undepreciated capital back into consumption goods at the same fixed rate to use for consumption. As in the illustrative model, a shock to the inter-temporal wedge D_t will generate a negative benchmark real interest rate in the low state.

First-order conditions of the household optimization problem:

$$(c_t^m)^{-\gamma} - \lambda_t = 0$$
$$\beta D_t \mathbb{E}_t (c_{t+1}^o)^{-\gamma} - \lambda_{t+1} = 0$$
$$-\lambda_t + \lambda_{t+1}(1 + r_t) = 0$$
$$-\chi (h_t^m)^{\frac{1}{\eta}} + \lambda_t w_t (1 - \tau_t) = 0$$
$$-\chi \beta D_t \mathbb{E}_t (h_{t+1}^o)^{\frac{1}{\eta}} + \lambda_{t+1} w_{t+1}(1 - \tau_{t+1}) = 0$$
$$-\lambda_t + \lambda_t r_t^k (1 - \tau_t) + \lambda_{t+1}(1 - \delta) = 0$$

Combining the first three yields the Euler equation:

$$(c_t^m)^{\gamma} = \beta D_t (1 + r_t) \mathbb{E}_t (c_{t+1}^o)^{\gamma}$$

Combining the first through fourth and second through fifth equations yields the labor supply conditions:

$$\chi (h_t^m)^{\frac{1}{\eta}} (c_t^m)^{\gamma} = w_t (1 - \tau_t)$$
$$\chi \mathbb{E}_t (h_{t+1}^o)^{\frac{1}{\eta}} \left[\mathbb{E}_t (c_{t+1}^o)^{-\gamma}\right]^{-1} = w_{t+1}(1 - \tau_{t+1})$$

Combining the third and sixth equations yields the no-arbitrage condition:

$$-\lambda_t + \lambda_t r_t^k (1 - \tau_t) + \lambda_{t+1}(1 - \delta) = 0$$
$$\lambda_t \left[1 - r_t^k (1 - \tau_t)\right] = \lambda_{t+1}(1 - \delta)$$
$$r_t^k (1 - \tau_t) = 1 - \left(\frac{1 - \delta}{1 + r_t}\right)$$

Under full depreciation, this reduces to: $r_t^k = \frac{1}{1-\tau_t}$.

Firms:
The production side of the economy now includes capital in the firm's production function. Thus, there are constant returns to scale in production and profits are zero in equilibrium.

$$\max_{L_t, K_t} Z_t = K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - r_t^k K_t$$

First-order conditions of the firm optimization problem:

$$w_t = (1 - \alpha) \frac{Y_t}{L_t}$$
$$r_t^k = \alpha \frac{Y_t}{K_t}$$
$$Z_t = 0$$

Firm ownership is irrelevant now as profits are zero.

Government:
The government's problem is the same as before, except tax revenues are now derived from distortionary income taxes on both labor wage and capital rental income. Government purchases and debt are given

exogenously in every period. The tax rate is determined endogenously to balance the budget.

$$\tau_t w_t L_t + \tau_t r_t^k K_t + \tau_t Z_t = (1 + r_{t-1}) B_{t-1}^g + G_t + B_t^g$$
$$\implies \tau_t Y_t = (1 + r_{t-1}) B_{t-1}^g + G_t + B_t^g$$

Equilibrium:

The flexible-price competitive equilibrium in this economy is defined as the sequence of endogenous variables $\{c_t^m, c_t^o, a_t^m, r_t, w_t, Y_t, h_t^m, h_t^o, L_t, \tau_t, K_t^s, K_t^d, r_t^k\}$ that satisfy the following equations, given exogenous $\{B_t^g, G_t, D_t\}$:

$$(c_t^m)^{-\gamma} = \beta D_t (1 + r_t) \mathbb{E}_t (c_{t+1}^o)^{-\gamma}$$
$$c_t^m = w_t h_t^m (1 - \tau_t) - a_t^m - K_t^s + r_t^k (1 - \tau_t) K_t^s$$
$$c_t^o = w_t h_t^o (1 - \tau_t) + (1 + r_{t-1}) a_{t-1}^m + (1 - \delta) K_{t-1}^s$$
$$\chi (h_t^m)^{\frac{1}{\eta}} (c_t^m)^{\gamma} = w_t (1 - \tau_t)$$
$$\chi \mathbb{E}_t (h_{t+1}^o)^{\frac{1}{\eta}} \left[\mathbb{E}_t (c_{t+1}^o)^{-\gamma}\right]^{-1} = w_{t+1}(1 - \tau_{t+1}) \Rightarrow \chi (h_t^o)^{\frac{1}{\eta}} (c_t^o)^{\gamma} = w_t (1 - \tau_t)$$
$$r_t^k (1 - \tau_t) = 1 - \left(\frac{1 - \delta}{1 + r_t}\right)$$
$$Y_t = \left(K_t^d\right)^{\alpha} L_t^{1-\alpha}$$
$$w_t = (1 - \alpha) \frac{Y_t}{L_t}$$
$$r_t^k = \alpha \frac{Y_t}{K_t^d}$$
$$\tau_t Y_t = (1 + r_{t-1}) B_{t-1}^g + G_t + B_t^g$$
$$L_t = h_t^m + h_t^o$$
$$a_t^m = B_t^g + K_t^s$$
$$K_t^s = K_t^d$$

C Appendix: Quantitative Life Cycle Model with Aggregate Uncertainty

First-order conditions of the household optimization problem:

$$u_n(c_{t+s-1}^s, n_{t+s-1}^s) = \beta^{s-1} \mathbb{E}_t D_{t+s-1} (-\chi_s) \left(n_{t+s-1}^s\right)^{\frac{1}{\eta}} = -\lambda_t^s (1 - \tau_t) w_t$$

$$u_c(c_{t+s-1}^s, n_{t+s-1}^s) = \beta^{s-1} \mathbb{E}_t D_{t+s-1} \left(c_{t+s-1}^s\right)^{-\gamma} = \lambda_t^s$$

$$-\lambda_t^s = (1 + r_{t+1}) \lambda_{t+1}^{s+1}$$

We can combine these to obtain the following intra-temporal labor-leisure condition and inter-temporal Euler equation, respectively:

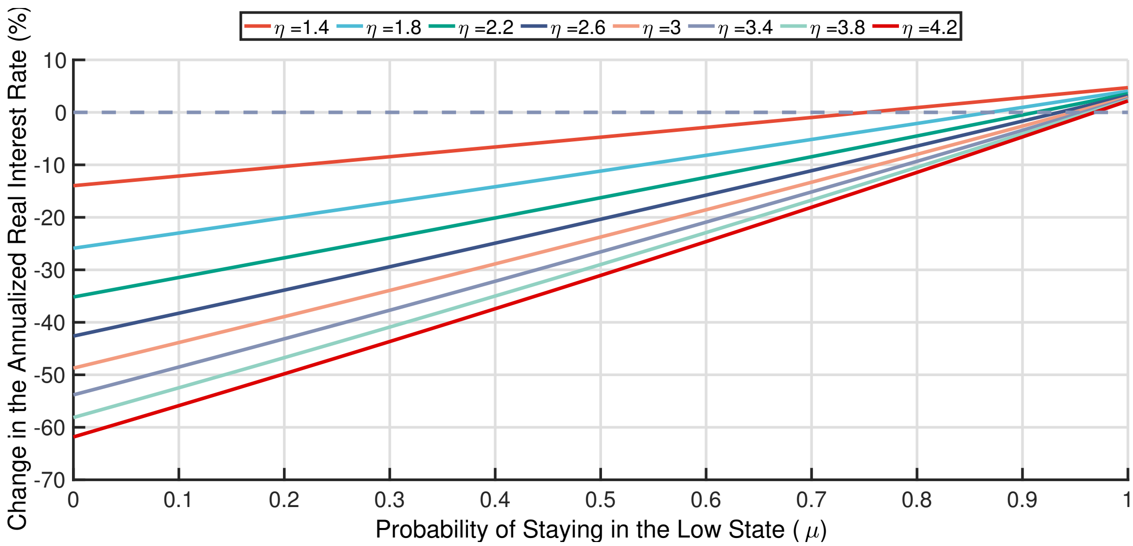
$$(c_t^s)^{\gamma} \chi_s (n_t^s)^{\frac{1}{\eta}} = (1 - \tau_t) w_t$$
$$D_t (c_t^s)^{-\gamma} = \beta \mathbb{E}_t (1 + r_{t+1}) D_{t+1} \left(c_{t+1}^{s+1}\right)^{-\gamma}$$

First-order conditions of the firm optimization problem:

$$w_t = (1 - \alpha) N_t^{-\alpha} K_t^{\alpha}$$

$$r_t = \alpha N_t^{1-\alpha} K_t^{\alpha-1} - \delta$$

Figure 12: Sensitivity analysis with respect to η: Effect of increasing debt on the low-state natural real interest rate



Source: Author's calculations.
Notes: This figure plots the change in the annualized real interest rate (in percent) against different probabilities of staying in the low state in the next period, for a range of ν_g. The computation holds the initial annual debt-to-GDP ratio fixed at 3 percent.

Government budget constraint:

$$\tau_t w_t N_t + B^g_{t+1} = (1+r_t)B^g_t + G_t + \frac{20}{60}b$$

Market clearing conditions:
In equilibrium, individual and aggregate behavior are consistent:

$$N_t = \Sigma_{s=1}^{60}\frac{n_t^s}{60}$$

$$K_{t+1} + B^g_{t+1} = \Sigma_{s=1}^{60}\frac{a_{t+1}^{s+1}}{60}$$

and the goods market clears:

$$N_t^{1-\alpha}K_t^{\alpha} = \Sigma_{s=1}^{60}\frac{c_t^s}{60} + K_{t+1} - (1-\delta)K_t + G_t$$

C.1 Non-stochastic Steady State

For $s = 1, \ldots, T-1$:

$$(c^s)^{\gamma}\chi_s(n^s)^{\frac{1}{\eta}} = (1-\tau)w \tag{2}$$

$$\left(\frac{(1+r)a^s + (1-\tau)wn^s - a^{s+1}}{(1+r)a^{s+1} + (1-\tau)wn^{s+1} - a^{s+2}}\right)^{-\gamma} = \beta(1+r) \tag{3}$$

For $s = T$:

$$(c^s)^{\gamma}\chi_s(n^s)^{\frac{1}{\eta}} = (1-\tau)w \tag{4}$$

$$\left(\frac{(1+r)a^s+(1-\tau)wn^s-a^{s+1}}{(1+r)a^{s+1}+b-a^{s+2}}\right)^{-\gamma}=\beta(1+r) \tag{5}$$

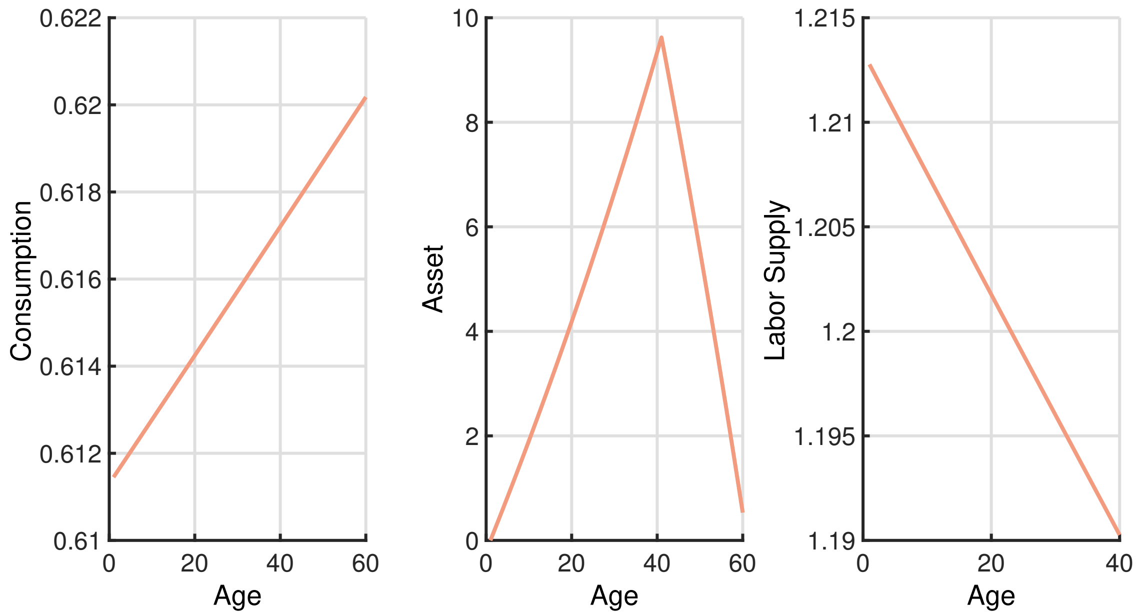
For $s=T+1,\ldots,T+R-1$:

$$\left(\frac{(1+r)a^s+b-a^{s+1}}{(1+r)a^{s+1}+b-a^{s+2}}\right)^{-\gamma}=\beta(1+r) \tag{6}$$

Since the households are born and die without assets, $a^{T+R+1}\equiv 0$. The equations (2) through (6) constitute a system of T+R-1+T = 2T+R-1 equations in the T+R-1+T = 2T+R-1 unknowns $\{a^s\}_{s=2}^{T+R}$ and $\{n^s\}_{s=1}^{T}$.

Figure 13 shows the distributions for consumption, asset, and labor supply in this non-stochastic steady state under the baseline calibration.

Figure 13: Distributions in the non-stochastic steady state



Source: Author's calculations.
Notes: This figure plots the distributions for consumption, asset, and labor supply in the non-stochastic business-as-usual steady state for the baseline calibration of the quantitative model provided in Table 3.

C.2 Linear Approximation

Log-linearization of the household first-order conditions around the business-as-usual steady state:

$$\frac{1}{\eta}\hat{n}_t^s=\hat{\lambda}_t^s-\frac{\tau}{1-\tau}\hat{\tau}_t+\hat{w}_t-\hat{D}_t,\quad s=1,\ldots,T$$

$$-\gamma\hat{c}_t^s=\hat{\lambda}_t^s-\hat{D}_t,\quad s=1,\ldots,T+R$$

$$\hat{\lambda}_t^s=\mathbb{E}_t\hat{\lambda}_{t+1}^{s+1}+\frac{r}{1+r}\mathbb{E}_t\hat{r}_{t+1},\quad s=1,\ldots,T+R-1$$

Log-linearization of the working household's budget constraint around the steady state for the 1-year-old household with $a_t^1\equiv 0$:

$$a^2\hat{a}_{t+1}^2 = -\tau wn^1\hat{\tau}_t + (1-\tau)wn^1\hat{w}_t + (1-\tau)wn^1\hat{n}_t^1 - c^1\hat{c}_t^1$$

and for $s = 2, \dots, T$:

$$a^{s+1}\hat{a}_{t+1}^{s+1} = (1+r)a^s\hat{a}_t^s + ra^s\hat{r}_t - \tau wn^s\hat{\tau}_t + (1-\tau)wn^s\hat{w}_t + (1-\tau)wn^s\hat{n}_t^s - c^s\hat{c}_t^s$$

Log-linearization of the retired household's budget constraint around the steady state for $s = T+1, \dots, T+R-1$:

$$a^{s+1}\hat{a}_{t+1}^{s+1} = (1+r)a^s\hat{a}_t^s + ra^s\hat{r}_t - c^s\hat{c}_t^s$$

and for $s = T+R$:

$$c^{T+R}\hat{c}_t^{T+R} = (1+r)a^{T+R}\hat{a}_t^{T+R} + ra^{T+R}\hat{r}_t$$

Thus, we have $T+R$ controls $\hat{c}_t^s$, $s = 1, \dots, T+R$, T controls $\hat{n}_t^s$, $s = 1, \dots, T$, $T+R$ costates $\hat{\lambda}_t^s$, $s = 1, \dots, T+R$ and $T+R-1$ predetermined variables $\hat{a}_t^s$, $s = 2, \dots, T+R$. We also have $(T)+(T+R)+(T+R-1)+(T+R) = 4T+3R-1$ equations.

We have three further endogenous variables $\hat{w}_t, \hat{r}_t, \hat{\tau}_t$. Log-linearization of the reward to each factor along with the market-clearing equation yields:

$$\hat{w}_t = -\alpha\Sigma_{s=1}^{40}\frac{n^s}{N}\frac{1}{60}\hat{n}_t^s + \alpha\left[\frac{A}{K}\Sigma_{s=1}^{60}\frac{a^s}{A}\frac{1}{60}\hat{a}_t^s - \frac{B^g}{K}\hat{B}_t^g\right]$$

$$\hat{r}_t = (1-\alpha)\Sigma_{s=1}^{40}\frac{n^s}{N}\frac{1}{60}\hat{n}_t^s - (1-\alpha)\left[\frac{A}{K}\Sigma_{s=1}^{60}\frac{a^s}{A}\frac{1}{60}\hat{a}_t^s - \frac{B^g}{K}\hat{B}_t^g\right]$$

Log-linearization of the government budget constraint:

$$\hat{\tau}_t = -\hat{w}_t - \Sigma_{s=1}^{40}\frac{n^s}{N}\frac{1}{60}\hat{n}_t^s - \frac{B^g}{\tau wn}\hat{B}_{t+1}^g + \frac{(1+r)B^g}{\tau wn}\hat{B}_t^g + \frac{rB^g}{\tau wn}\hat{r}_t + \frac{G}{\tau wn}\hat{G}_t$$

These constitute three further equations in three endogenous variables $\hat{w}_t, \hat{r}_t, \hat{\tau}_t$.

Finally, we need the law of motion for the three exogenous variables $\hat{B}_t^g$, $\hat{G}_t$, $\hat{D}_t$ to close the model. For $\mathbf{z}_t = \left[\hat{D}_t, \hat{G}_t, \hat{B}_t^g\right]'$, this is presented as:

$$\mathbb{E}_t\mathbf{z}_{t+1} = \Pi\mathbf{z}_t$$

C.3 Solution Algorithm

Step 1: Express as a Linear Rational Expectations Model

It is convenient to express this system of equations in the form:

$$C_u\mathbf{u}_t = C_{x\lambda}\begin{bmatrix}\mathbf{x}_t\\ \boldsymbol{\lambda}_t\end{bmatrix} + C_z\mathbf{z}_t$$

$$D_{x\lambda}\mathbb{E}_t\begin{bmatrix}\mathbf{x}_{t+1}\\ \boldsymbol{\lambda}_{t+1}\end{bmatrix} + F_{x\lambda}\begin{bmatrix}\mathbf{x}_t\\ \boldsymbol{\lambda}_t\end{bmatrix} = D_u\mathbb{E}_t\mathbf{u}_{t+1} + F_u\mathbf{u}_t + D_z\mathbb{E}_t\mathbf{z}_{t+1} + F_z\mathbf{z}_t$$

where, I define:

$$\mathbf{u}_t = \left[\hat{c}_t^1, \hat{c}_t^2 \dots, \hat{c}_t^{60}, \hat{n}_t^1, \hat{n}_t^2, \dots, \hat{n}_t^{40}, \hat{r}_t, \hat{w}_t, \hat{\tau}_t\right]'$$

$$\mathbf{x}_t = \left[\hat{a}_t^2, \hat{a}_t^3, \dots, \hat{a}_t^{60}\right]'$$

$$\boldsymbol{\lambda}_t = \left[\hat{\lambda}_t^1, \hat{\lambda}_t^2, \dots, \hat{\lambda}_t^{59}\right]'$$

$$\mathbf{z}_t = \left[\hat{D}_t, \hat{G}_t, \hat{B}_t^g\right]'$$

Step 2: Solve the business-as-usual state solution
Solving this system of equations yields a solution of the form:

$$\mathbf{x}_{t+1} = L_x^x \mathbf{x}_t + L_z^x \mathbf{z}_t$$

$$\mathbf{u}_t = L_x^u \mathbf{x}_t + L_z^u \mathbf{z}_t$$

$$\boldsymbol{\lambda}_t = L_x^l \mathbf{x}_t + L_z^l \mathbf{z}_t$$

Step 3: Solve the low-state solution
In the low state, the equations are the same as in the business-as-usual state. The only difference is that the expectation of the variables that are not predetermined at time t depends on the probability of benchmark real rate normalization. In particular,

$$\begin{aligned}
\mathbb{E}_t \mathbf{u}_{t+1} &= \mu \mathbb{E}_t \mathbf{u}_{t+1|\text{low state}} + (1-\mu)\mathbb{E}_t \mathbf{u}_{t+1|\text{business-as-usual state}} \\
&= \mu \mathbb{E}_t \mathbf{u}_{t+1|\text{low state}} + (1-\mu) L_x^u \mathbb{E}_t \mathbf{x}_{t+1|\text{low state}} + (1-\mu) L_z^u \mathbb{E}_t \mathbf{z}_\Gamma \\
\mathbb{E}_t \boldsymbol{\lambda}_{t+1} &= \mu \mathbb{E}_t \boldsymbol{\lambda}_{t+1|\text{low state}} + (1-\mu)\mathbb{E}_t \boldsymbol{\lambda}_{t+1|\text{business-as-usual state}} \\
&= \mu \mathbb{E}_t \boldsymbol{\lambda}_{t+1|\text{low state}} + (1-\mu) L_x^l \mathbb{E}_t \mathbf{x}_{t+1|\text{low state}} + (1-\mu) L_z^l \mathbb{E}_t \mathbf{z}_\Gamma
\end{aligned}$$

This requires modifying the input matrices slightly and resolving the entire system of equations using the new modified matrix system.[19]

Again, the solution will be of the form:

$$\mathbf{x}_{t+1} = L_x^x \mathbf{x}_t + L_z^x \mathbf{z}_t$$

$$\mathbf{u}_t = L_x^u \mathbf{x}_t + L_z^u \mathbf{z}_t$$

$$\boldsymbol{\lambda}_t = L_x^l \mathbf{x}_t + L_z^l \mathbf{z}_t$$

From this, we can extract the coefficient of the shocks to derive the effect of public debt on the natural real interest rate (and other macroeconomics variables) in the economy.

The solution algorithm is based on a generalized Schur decomposition as described in Klein (2000). Replication codes are available on my website at www.vaishaligarga.com.

C.3.1 Input matrices for numerical solution

We partition the entire system of equations into two: contemporary system and dynamic system, as discussed in the general solution method above.

Contemporary Equations:

$$\frac{1}{\eta}\hat{n}_t^s + \frac{\tau}{1-\tau}\hat{\tau}_t - \hat{w}_t = \hat{\lambda}_t^s - \hat{D}_t, \quad s = 1, \ldots, 40$$

$$-\gamma \hat{c}_t^s = \hat{\lambda}_t^s - \hat{D}_t, \quad s = 1, \ldots, 59$$

$$c^{60}\hat{c}_t^{60} - r a^{60}\hat{r}_t = (1+r)a^{60}\hat{a}_t^{60}$$

$$\hat{w}_t + \alpha \Sigma_{s=1}^{40} \frac{n^s}{N}\frac{1}{60}\hat{n}_t^s = \alpha\left[\Sigma_{s=2}^{60}\frac{a^s}{K}\frac{1}{60}\hat{a}_t^s - \frac{B^g}{K}\hat{B}_t^g\right]$$

[19]The linearized shocks are zero in the business-as-usual state, because I linearize the model around the "final" business-as-usual steady state. Thus, I can eliminate $\mathbf{z}_t$ from the business-as-usual state rational expectations system, thereby also eliminating the need for defining a law of motion for it. See Appendix C.3.2 for further details.

$$\hat{r}_t - (1-\alpha)\Sigma_{s=1}^{40}\frac{n^s}{N}\frac{1}{60}\hat{n}_t^s = -(1-\alpha)\left[\Sigma_{s=2}^{60}\frac{a^s}{K}\frac{1}{60}\hat{a}_t^s - \frac{B^g}{K}\hat{B}_t^g\right]$$

$$\hat{\tau}_t + \hat{w}_t + \Sigma_{s=1}^{40}\frac{n^s}{N}\frac{1}{60}\hat{n}_t^s - \frac{rB^g}{\tau wn}\hat{r}_t = -\frac{B^g}{\tau wn}\hat{B}_{t+1}^g + \frac{(1+r)B^g}{\tau wn}\hat{B}_t^g + \frac{G}{\tau wn}\hat{G}_t$$

Dynamic Equations:

$$\hat{\lambda}_t^s - \mathbb{E}_t\hat{\lambda}_{t+1}^{s+1} = \frac{r}{1+r}\mathbb{E}_t\hat{r}_{t+1}, \quad s = 1, \ldots, 58$$

$$\hat{\lambda}_t^{59} = -\gamma\mathbb{E}_t\hat{c}_{t+1}^{60} + \frac{r}{1+r}\mathbb{E}_t\hat{r}_{t+1}$$

$$a^2\hat{a}_{t+1}^2 = -\tau wn^1\hat{\tau}_t + (1-\tau)wn^1\hat{w}_t + (1-\tau)wn^1\hat{n}_t^1 - c^1\hat{c}_t^1$$

$$a^{s+1}\hat{a}_{t+1}^{s+1} - (1+r)a^s\hat{a}_t^s = ra^s\hat{r}_t - \tau wn^s\hat{\tau}_t + (1-\tau)wn^s\hat{w}_t + (1-\tau)wn^s\hat{n}_t^s - c^s\hat{c}_t^s,$$

s = 2, ..., 40

$$a^{s+1}\hat{a}_{t+1}^{s+1} - (1+r)a^s\hat{a}_t^s = ra^s\hat{r}_t - c^s\hat{c}_t^s,$$

s = 41, ..., 59

So far, this is a system of 222 equations in 225 unknowns. We need three more equations for the exogenous variables that constitute $\mathbf{z}_t$:

$$\mathbb{E}_t\mathbf{z}_{t+1} = \Pi\mathbf{z}_t$$

$$\Pi = \begin{bmatrix} \mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Stability requires that the eigenvalues of the matrix Π lie within the unit circle.

C.3.2 Business-as-usual state:

The contemporary equation system is characterized by matrices C_u, $C_{x\lambda}$, and C_z defined as follows:

$$A_{11} = \mathbf{0}$$

$$A_{12} = \begin{bmatrix} \frac{1}{\eta} & 0 & \cdots & 0 \\ 0 & \frac{1}{\eta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\eta} \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} 0 & -1 & \frac{\tau}{1-\tau} \\ 0 & -1 & \frac{\tau}{1-\tau} \\ \vdots & \vdots & \vdots \\ 0 & -1 & \frac{\tau}{1-\tau} \end{bmatrix}$$

and the 60×60 submatrix A_{21}, 60×40 submatrix A_{22}, and 60×3 submatrix A_{23} as follows:

$$A_{22} = \mathbf{0}$$

$$A_{21} = \begin{bmatrix} -\gamma & 0 & \cdots & 0 & 0 \\ 0 & -\gamma & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\gamma & 0 \\ 0 & 0 & \cdots & 0 & c^{60} \end{bmatrix}$$

$$A_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ -ra^{60} & 0 & 0 \end{bmatrix}$$

and the 3×60 submatrix A_{31}, 3×40 submatrix A_{32}, and 3×3 submatrix A_{33} as follows:

$$A_{31} = \mathbf{0}$$

$$A_{32} = \begin{bmatrix} \alpha \frac{n^1}{N} \frac{1}{60} & \alpha \frac{n^2}{N} \frac{1}{60} & \cdots & \alpha \frac{n^{40}}{N} \frac{1}{60} \\ -(1-\alpha) \frac{n^1}{N} \frac{1}{60} & -(1-\alpha) \frac{n^2}{N} \frac{1}{60} & \cdots & -(1-\alpha) \frac{n^{40}}{N} \frac{1}{60} \\ \frac{n^1}{N} \frac{1}{60} & \frac{n^2}{N} \frac{1}{60} & \cdots & \frac{n^{40}}{N} \frac{1}{60} \end{bmatrix}$$

$$A_{33} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -\frac{rB^g}{\tau w N} & 1 & 1 \end{bmatrix}$$

$$C_{x\lambda} = \begin{bmatrix} C_{x\lambda,11} & C_{x\lambda,12} \\ C_{x\lambda,21} & C_{x\lambda,22} \\ C_{x\lambda,31} & C_{x\lambda,32} \end{bmatrix}$$

with 40×59 submatrix $C_{x\lambda,11}$, 40×59 submatrix $C_{x\lambda,12}$, 60×59 submatrix $C_{x\lambda,21}$, 60×59 submatrix $C_{x\lambda,22}$, 3×59 submatrix $C_{x\lambda,31}$, and 3×59 submatrix $C_{x\lambda,32}$ defined as:

$$C_{x\lambda,11} = C_{x\lambda,32} = \mathbf{0}$$

$$C_{x\lambda,12} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \end{bmatrix}$$

$$C_{x\lambda,21} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \ldots & 0 & (1+r)a^{60} \end{bmatrix}$$

$$C_{x\lambda,22} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

$$C_{x\lambda,31} = \begin{bmatrix} \alpha \frac{a^2}{K} \frac{1}{60} & \alpha \frac{a^3}{K} \frac{1}{60} & \cdots & \alpha \frac{a^{60}}{K} \frac{1}{60} \\ -(1-\alpha) \frac{a^2}{K} \frac{1}{60} & -(1-\alpha) \frac{a^3}{K} \frac{1}{60} & \cdots & -(1-\alpha) \frac{a^{60}}{K} \frac{1}{60} \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The dynamic equation system is characterized by matrices $D_{x\lambda}$, $F_{x\lambda}$, D_u, and F_u defined as follows:

$$D_{x\lambda} = \begin{bmatrix} D_{x\lambda,11} & D_{x\lambda,12} \\ D_{x\lambda,21} & D_{x\lambda,22} \end{bmatrix}$$

where $D_{x\lambda,ij}$ are 59×59 submatrices defined as:

$$D_{x\lambda,11} = D_{x\lambda,22} = \mathbf{0}$$

$$D_{x\lambda,12} = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$D_{x\lambda,21} = \begin{bmatrix} a^2 & 0 & \cdots & 0 \\ 0 & a^3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^{60} \end{bmatrix}$$

$$F_{x\lambda} = \begin{bmatrix} F_{x\lambda,11} & F_{x\lambda,12} \\ F_{x\lambda,21} & F_{x\lambda,22} \end{bmatrix}$$

where $F_{x\lambda,ij}$ are 59×59 submatrices defined as:

$$F_{x\lambda,11} = F_{x\lambda,22} = \mathbf{0}$$

$$F_{x\lambda,12} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$F_{x\lambda,21} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -(1+r)a^2 & 0 & \cdots & 0 & 0 \\ 0 & -(1+r)a^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -(1+r)a^{59} & 0 \end{bmatrix}$$

$$D_u = \begin{bmatrix} D_{u,11} & D_{u,12} & D_{u,13} \\ D_{u,21} & D_{u,22} & D_{u,23} \end{bmatrix}$$

where $D_{u,11}, D_{u,21}$ are 59×60 submatrices, $D_{u,12}, D_{u,22}$ are 59×40 submatrices, and $D_{u,13}, D_{u,23}$ are 59×3 submatrices defined as:

$$D_{u,12} = D_{u,21} = D_{u,22} = D_{u,23} = \mathbf{0}$$

$$D_{u,11} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -\gamma \end{bmatrix}$$

$$D_{u,13} = \begin{bmatrix} \frac{r}{1+r} & 0 & \cdots & 0 \\ \frac{r}{1+r} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{r}{1+r} & 0 & \cdots & 0 \end{bmatrix}$$

$$F_u = \begin{bmatrix} F_{u,11} & F_{u,12} & F_{u,13} \\ F_{u,21} & F_{u,22} & F_{u,23} \end{bmatrix}$$

where $F_{u,11}, F_{u,21}$ are 59×60 submatrices, $F_{u,12}, F_{u,22}$ are 59×40 submatrices, and $F_{u,13}, F_{u,23}$ are 59×3 submatrices defined as:

$$F_{u,11} = F_{u,12} = F_{u,13} = \mathbf{0}$$

$$F_{u,21} = \begin{bmatrix} -c^1 & 0 & \cdots & 0 & 0 \\ 0 & -c^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -c^{59} & 0 \end{bmatrix}$$

$$F_{u,22} = \begin{bmatrix} (1-\tau)wn^1 & 0 & \cdots & 0 \\ 0 & (1-\tau)wn^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (1-\tau)wn^{40} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$F_{u,23} = \begin{bmatrix} 0 & (1-\tau)wn^1 & -\tau wn^1 \\ ra^2 & (1-\tau)wn^2 & -\tau wn^2 \\ ra^3 & (1-\tau)wn^3 & -\tau wn^3 \\ \vdots & \vdots & \vdots \\ ra^{40} & (1-\tau)wn^{40} & -\tau wn^{40} \\ ra^{41} & 0 & 0 \\ \vdots & \vdots & \vdots \\ ra^{59} & 0 & 0 \end{bmatrix}$$

C.3.3 Low state:

The contemporary equation system is again characterized by matrices C_u, $C_{x\lambda}$ defined above and matrix C_z defined as follows:

$$C_z = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha\frac{B^g}{K} \\ 0 & 0 & (1-\alpha)\frac{B^g}{K} \\ 0 & \frac{G}{\tau wN} & \frac{(1+r)B^g}{\tau wN} - \frac{B^g}{\tau wN} \end{bmatrix}$$

Here, I use $\hat{B}^g_{t+1} = \hat{B}^g_t$ in pinning down the final coefficient of this matrix.

The dynamic equation system is characterized by matrices $D_{x\lambda}$, $F_{x\lambda}$, D_u, F_u, D_z, and F_z as follows:

$$\begin{bmatrix} D_{11}^{x\lambda} + D_{12}^{x\lambda}(1-\mu)L_x^{\lambda} & D_{12}^{x\lambda}\mu \\ D_{21}^{x\lambda} + D_{22}^{x\lambda}(1-\mu)L_x^{\lambda} & D_{22}^{x\lambda}\mu \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \mathbf{x}_{t+1} \\ \boldsymbol{\lambda}_{t+1} \end{bmatrix} + F_{x\lambda} \begin{bmatrix} \mathbf{x}_t \\ \boldsymbol{\lambda}_t \end{bmatrix} = D_u \mu \mathbb{E}_t \mathbf{u}_{t+1} + D_u(1-\mu)L_x^u \mathbf{x}_{t+1} + F_u \mathbf{u}_t + D_z \mu \mathbb{E}_t \mathbf{z}_{t+1} + F_z \mathbf{z}_t$$

where,

$$D_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

$$F_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

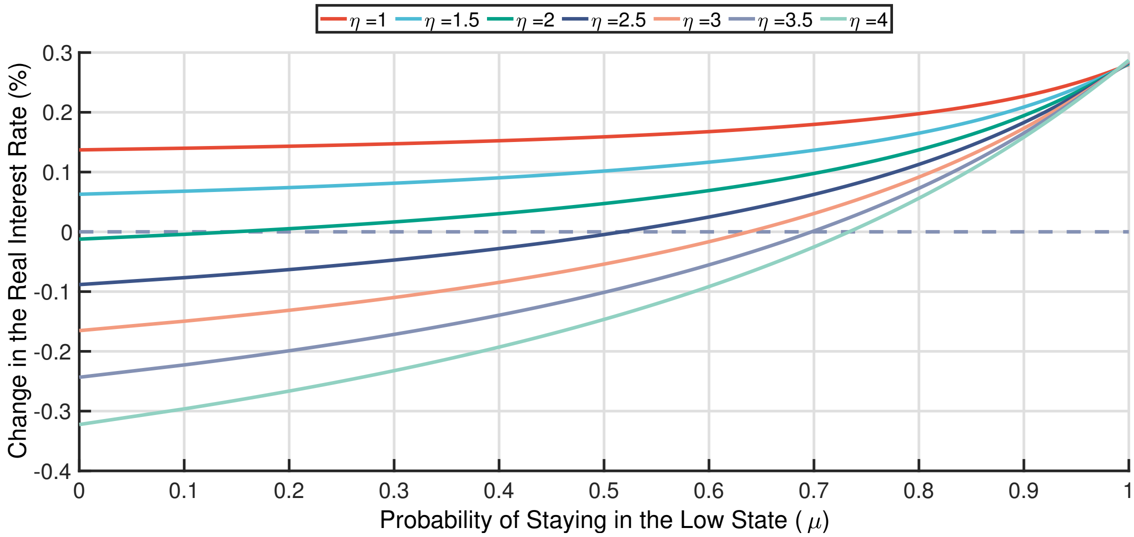
Table 5: Debt-to-GDP threshold corresponding to different μ values

μ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
expected duration	1.11	1.25	1.42	1.66	2	2.5	3.33	5	10
$\frac{B^g}{Y}^*$(%)	100	130	160	200	250	310	380	490	>496

Table 6: Sensitivity of μ^* with respect to η (conditional on initial debt-to-GDP ratio at 110 percent)

η	1.5	2	2.5	3	3.5	4
μ^*	0	0.14	0.51	0.63	0.69	0.73

Figure 14: Sensitivity analysis with respect to η: Effect of increasing debt on the low-state natural real interest rate



Source: Author's calculations.
Notes: This figure plots the change in the real interest rate (in percent) against different probabilities of staying in the low state in the next period, for a range of η. The computation holds the initial debt-to-GDP ratio fixed at 110 percent.

C.4 Derivation: Expected Duration of Low State

Given the two-state Markov process for the shock driving the benchmark real rate:

$$\begin{aligned}\text{Expected duration of low state} &= (1-\mu) + \mu(1-\mu)2 + \mu^2(1-\mu)3 + \ldots \\ &= (1-\mu)\left[1 + 2\mu + 3\mu^2 + \ldots\right] \\ &= (1-\mu)\left[\frac{1}{1-\mu} + \frac{\mu}{(1-\mu)^2}\right] \\ &= (1-\mu)\left[\frac{(1-\mu)+\mu}{(1-\mu)^2}\right] \\ &= \frac{1}{1-\mu}\end{aligned}$$

D Appendix: Extension with Sovereign Default Risk

Another issue with running up massive stocks debt is its potential unsustainability in the state of the world where the benchmark real rates normalize. The possibility of a sovereign debt crisis makes government debt risky, thereby increasing the real interest rate that needs to be paid on it to compensate investors for the risk of holding it.

Sovereign debt is a complex issue with several important considerations. In this section, I incorporate it in a simple way into my model and show how it creates an additional mechanism through which debt is costly in terms of output in the business-as-usual state.

Additional Model Elements

Consider now that government debt is a risky asset that pays return r_t^v, but with probability ρ_{t+1} it will not be repaid in the future. This gives rise to a no-arbitrage condition between the risky rate and the risk-free rate:

$$1 + r_t = (1 + r_t^v)\mathbb{E}_t(1 - \rho_{t+1})$$

Thus, with the presence of sovereign default risk, the government budget constraint changes—the relevant real interest rate is now the risky rate:

$$\tau_t Y_t = \left[\frac{1 + r_{t-1}}{1 - \rho_t} - 1\right] B_{t-1}^g + G_t - B_t^g$$

We immediately see that a higher default risk implies a higher risk premium, which in turn implies a higher tax rate for any given level of output.

The no-arbitrage condition between government debt and capital also changes so that the no-arbitrage is with respect to the risky real interest rate rather than the safe real interest rate. The old-age budget constraint also now features the risky rate instead of the safe rate.

Assumption 5. *Default Risk: The probability of non-repayment is given by:*

$$\rho_t = \rho\left(B_t^g\right) = d_0 exp\left(d_1 B_t^g\right), \quad t \geq \Gamma$$

$$\rho_t = 0, \quad t < \Gamma$$

where d_0 is the probability of default when there is no government debt and d_1 is the elasticity of the probability of default with respect to the change in the stock of government debt.

I assume that the probability of non-repayment is positive only in the business-as-usual state, because there is no reason for the government to default on its payment in the low state, where real rates are negative. I also assume that the probability depends positively on the stock of government debt.[20] To complete the model, I need to specify what happens in the economy if default actually occurs. Following the literature on sovereign default, I assume that default is costly in terms of output.

Assumption 6. *Occurence of Default: In the event of a default on government debt,*

$$Y_t = Y_t - d_g \mathbb{W}^*$$

where d_g is the output cost of default.

d_g is a reduced form parameter for the output cost of sovereign default.

To illustrate the results, I attach the following illustrative parameterization to the new parameters: $d_0 = 0.05$, $d_1 = 1.5$, and $d_g = 0$. I set d_g to zero to get the most conservative results. But it will become clear shortly that as d_g increases, the results regarding the perverse effect will become stronger.

Main Results

Figure 15 is the analog of Figure 6. A comparison of the two figures shows that while the effect of debt depends on μ in similar ways in both models, the threshold μ^* in the model with default is higher. This is because government debt imposes an additional cost in the business-as-usual state by imposing a risk premium on government debt, which necessitates higher distortionary taxation to service the debt. This results in a larger crowding out of labor supply and productive capital.

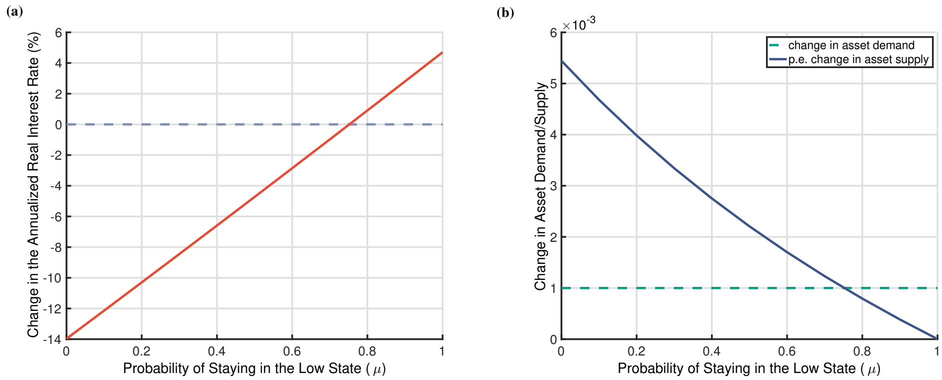
Similarly, Figure 16 is the analog of Figure 7. The threshold debt-to-GDP ratio in the model with default risk is lower than in the model without.

Sensitivity Analysis with Respect to η

As in the baseline model without default, with higher η, the perverse expectations channel will be stronger for any given μ. Therefore, the threshold μ^* is (weakly) increasing in η, as shown in Figure 17.

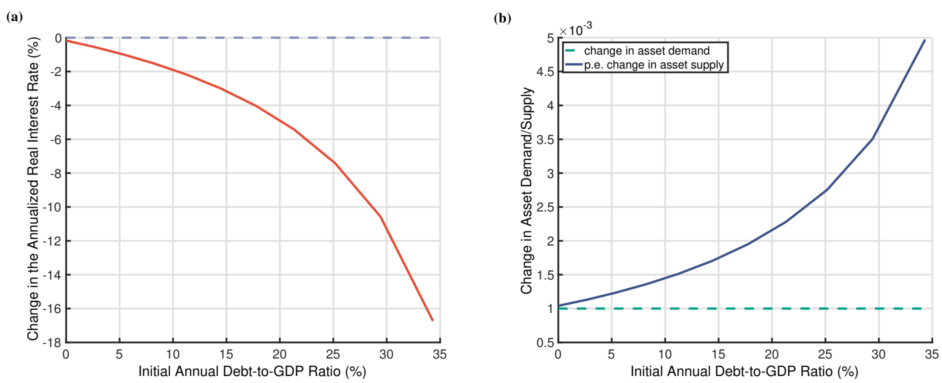
[20] This is motivated from the literature on sovereign default. Reinhart and Rogoff (2010) show that a crisis is more likely to occur as government debt outstanding increases. D'Erasmo and Mendoza (2016) show theoretically that default on domestic government debt is more likely when debt is larger and tax revenue is smaller.

Figure 15: Effect of increasing debt on the low-state natural real interest rate



Source: Author's calculations.
Notes: This figure plots the change in the annualized real interest rate (in percent) (panel [a]) and the change in the market for loanable funds (panel [b]), against different probabilities of staying in the low state in the next period, ranging from 0 to 1. The ratio is computed based on the annualized equilibrium output in the initial constant solution of the business-as-usual state.

Figure 16: Non-monotonicity of change in natural real interest rate with respect to the initial debt-to-GDP ratio



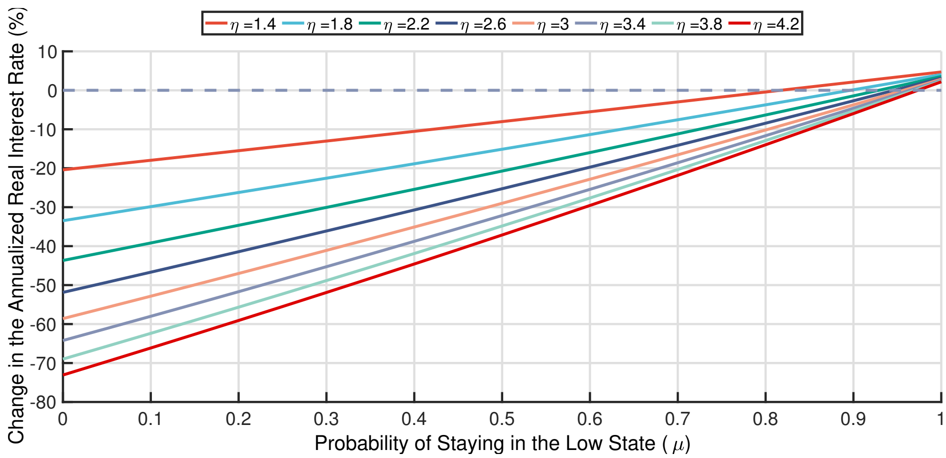
Source: Author's calculations.
Notes: This figure plots the change in the annualized real interest rate (in percent) (panel [a]) and the change in the market for loanable funds (panel [b]), against different initial debt-to-GDP (annual) ratios. The ratio is computed based on the annualized equilibrium output in the initial constant solution of the business-as-usual state. The computation holds the probability μ, in the stochastic process for the inter-temporal wedge shock, fixed at 0.7.

Comparing Table 7 here to Table 2 in the main text clearly reveals that the implicit output cost of debt in the business-as-usual state is higher in the model with default risk than without.

E Case in Point: Japan

The Japanese economy has experienced a seemingly secular decline in the real interest rate over the past three decades. Over the same period, the government has, in response, nearly quadrupled its debt-to-GDP

Figure 17: Sensitivity analysis with respect to η: Effect of increasing debt on the low-state natural real interest rate



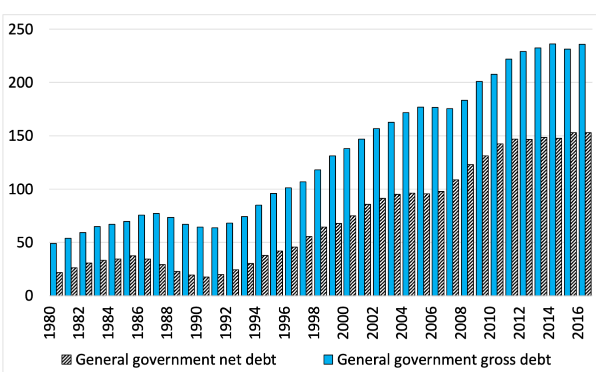
Source: Author's calculations.
Notes: This figure plots the change in the annualized real interest rate (in percent) against different probabilities of staying in the low state in the next period for a range of ν_g.

Table 7: Sensitivity analysis with respect to η

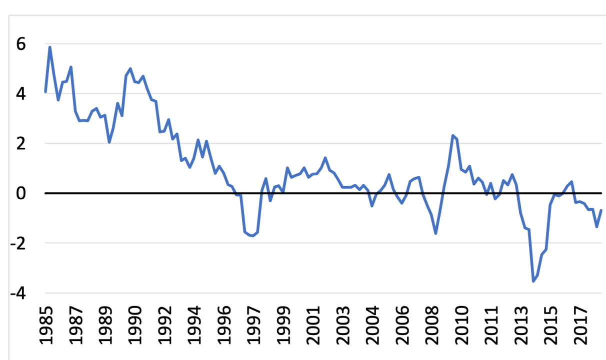
η	1.4	1.8	2.2	2.6	3	3.4
implicit ν_g	29.615	31.429	32.781	33.828	34.662	35.342
μ^*	0.8	0.85	0.9	0.9	0.95	0.95

ratio, increasing it from 50 percent to 230 percent (Figure 18). But the economy remains stagnated, with low real rates. The hypothesis in this paper suggests that this is because people are highly aware and worried about the potential burden of the extremely high debt in the event that real rates rise in the future.

Figure 18: Japan's debt-to-GDP ratio has quadrupled, yet real rates remain unchanged



(a) Debt-to-GDP ratio in Japan



(b) Real interest rates in Japan

Survey evidence from Japan provides suggestive evidence in support of this hypothesis. According to a household survey conducted by the Japanese Cabinet Office in 2016, an increasing number of Japanese "feel worries or anxieties in everyday life" and around a third of them report "fiscal balance" as the reason for these anxieties (Figure 19). Further, there is evidence that high debt is often associated with expectation of future tax increases and potential unsustainability of debt. Figure 20, taken from Kobayashi and Ueda (2017), shows the number of occurences of specific phrases in the morning and evening editions of the *Nihon Keizai Shinbun*, Japan's financial newspaper, from the years 1981 through 2016. The count suggests that the frequency of use of "fiscal failure (*zaisei hatan*)" or "fiscal crisis (*zaisei kiki*)," phrases that reflect a fiscal imbalance, has increased continuously since 1981. Moreover, a combination of the phrases "tax increases (*zozei*)" and either "fiscal failure" or "default" has been used more and more frequently as well. Masayuki (2017) provides further survey evidence showing that there is considerable uncertainty over the future course of the social security and tax systems in Japan, and that this uncertainty suppresses consumption.

Figure 19: Concerns about government debt

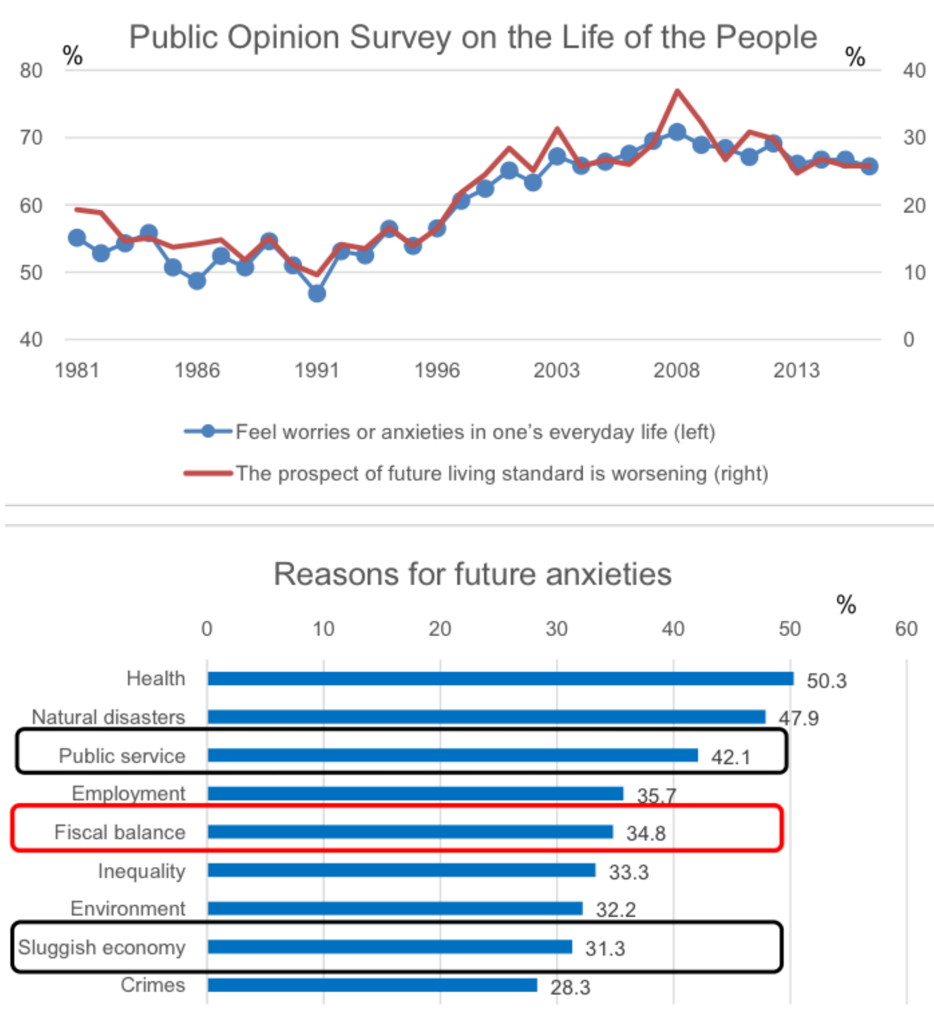


Figure 20: The number of occurrences of specific words in newspaper