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Sovereign Default Risk and Uncertainty Premia

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Abstract:
This paper studies how foreign investors' concerns about model misspecification affect sovereign bond spreads. We develop a general equilibrium model of sovereign debt with endogenous default wherein investors fear that the probability model of the underlying state of the borrowing economy is misspecified. Consequently, investors demand higher returns on their bond holdings to compensate for the default risk in the context of uncertainty. In contrast with the existing literature on sovereign default, we explain the bond spreads dynamics observed in the data as well as other business cycle features for Argentina, while preserving the default frequency at historical low levels.

Keywords: sovereign debt, default risk, model uncertainty, robust control

JEL Classifications: D81, E21, E32, E43, E34

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1 Introduction

Sovereign defaults, or debt crises in general, are a pervasive economic phenomenon, especially among emerging economies. Recent defaults by Russia (in 1998), Ecuador (in 1999) and Argentina (in 2001), and the current debt crises of Greece have put sovereign default issues at the forefront of economic policy discussion. A key aspect of default events, or rather risk of defaults, is that bond holders, by forecasting this event (and further contingencies regarding the debt restructuring process), require compensation for bearing this risk, which, at the same time, hinders the access to credit of the borrowing economies. Thus, high and volatile bond spreads translate into high and volatile borrowing costs for these economies. Therefore, constructing economic models that can both generate these default events and provide accurate predictions in terms of pricing, is key.

As is the case in most of the asset pricing literature, the literature on defaultable debt follows the rational expectations paradigm; lenders fully trust the single probability model governing the state of the economy and are not concerned with any source of potential misspecification. It is well documented that economic models using this paradigm face difficulties when confronted with the asset prices data. The case of defaultable debt (either corporate or sovereign) is not an exception. In this case, models are typically unable to account for the observed dynamics in the bond spreads, while preserving the default frequency at historical levels.[1] This paper tackles this "pricing puzzle"—while also accounting for other empirical regularities of emerging economies—by studying how lenders' desire to make decisions that are robust to model misspecification affects equilibrium prices and allocations, in an otherwise standard general equilibrium model of sovereign default.[2]

In this paper we adapt the seminal general equilibrium model of Eaton and Gersovitz (1981) by introducing lenders that distrust the probability model governing the evolution of the state of the borrowing economy and want to guard themselves against misspecification errors in it. In the model, a borrower (for example, an emerging economy) can trade one-period discount bonds with international lenders in financial markets. Debt repayments cannot be enforced and the emerging economy may decide to default. Lenders in equilibrium anticipate the default strategies of the emerging economies and demand higher returns on their sovereign bond holdings to compensate for the default risk. In case of default, the

[1] This phenomenon is not limited to the sovereign debt literature, since it is also well documented in the corporate debt literature; see Huang and Huang (2003) and Elton, Gruber, Agrawal, and Mann (2001), for example.

[2] For a summary of the empirical regularities in emerging economies, see, for example, Neumeyer and Perri (2005) and Uribe and Yue (2006).

economy is temporarily excluded from financial markets and suffers a direct output cost. In this setting, we show how lenders' desire to make decisions that are robust to misspecification of the conditional probability of the borrower's endowment alters the returns on sovereign bond holdings.[3]

The novelty in our paper comes from the fact that lenders are uncertainty averse in the sense that they are unwilling or unable to form a unique probability distribution or probability model for the endowment of the borrower, and at the same time they dislike making decisions in the context of alternative probability models. Indeed, lenders in our model acknowledge that the stochastic process of the borrower's endowment may be misspecified, and would like to make decisions that are robust to such misspecification. To express these fears about model misspecification, following Hansen and Sargent (2005) we endow lenders with multiplier preferences.[4,5] In this context, the lenders somehow construct a conditional probability distribution (referred to as "approximating probability distribution" or "approximating model") for the endowment of the borrower, but suspect it can be misspecified and hence surround it with alternative probability distributions that are statistical perturbations of it. To make choices that perform well over this set of probability distributions, the lender *acts as if* contemplating a conditional worst-case probability distribution that is distorted relative to his approximating one. This distorted conditional probability distribution is determined by minimizing the lender's objective function, while remaining statistically indistinguishable from the approximating probability distribution at some confidence levels. It therefore arises from perturbing the approximating density by slanting probability toward the states associated with low utility. In our model, these low-utility states for the lender coincide with those in which the borrower defaults on its debt.

The assumption about lenders' concerns about model misspecification is intended to capture the fact that foreign lenders may distrust their statistical model used to predict relevant macroeconomic variables of the emerging economy (that is, the borrower). In addition,

[3]By following Eaton and Gersovitz (1981), we abstract from transaction costs, liquidity restrictions, and other frictions that may affect the real return on sovereign bond holdings.

[4]In a wide class of environments, the utility recursion with multiplier preferences can be reinterpreted in terms of Epstein-Zin utility formulation, beyond the standard log period payoff specification. In such a case, the typical probability distortion through which the agent's uncertainty aversion is manifested would take the form of a risk-sensitive adjustment used to evaluate future streams of consumption. We view the contributions of model uncertainty and risk aversion not as mutually exclusive but rather as complementary, in line with Barillas, Hansen, and Sargent (2009) among others. In our framework, this apparent observational equivalence, however, does not apply, because the lender contemplates perturbations only to the probability model governing the evolution of the borrower's endowment, and not to the probability distribution of reentry to financial markets, which is assumed to be fully trusted.

[5]Axiomatic foundations for this class of preferences have been provided by Strzalecki (2011).

foreign lenders are aware of the limited availability of reliable official data, measurement errors, and lags in the release of the official statistics together with subsequent revisions.[6,7] These arguments are aligned with the suggested view of putting econometricians and agents in a position with identical information, and limitations on their ability to estimate statistical models.

The main result of our paper is that by introducing lenders' fears about model misspecification, our calibration matches the high, volatile, and typically countercyclical bond spreads observed in the data for the Argentinean economy, together with standard business cycle features, while keeping the default frequency at historical levels. At the same time, our model can account for the average risk-free rate observed in the data; model uncertainty in our economy does not alter the risk-free rate. It is worth pointing out that in the simulations we also find that under plausible values of the parameters, risk aversion alone on the lenders' side with time-separable preferences is not sufficient to generate the observed risk premia. Also, as the degree of lenders' risk aversion increases, the average net risk-free rate declines, eventually to negative levels.

In addition to studying the level, volatility, and countercyclicality of the bond spreads, we document and study higher moments, such as skewness and kurtosis, and different quantiles for the spreads. We believe that these moments are crucial for depicting the behavior of spreads, especially in economies such as the Argentinean one, because they contain information about their tail behavior, in particular about the upper tail, which accounts for the spreads in times of debt crises.

First, we document that spreads in the Argentinean economy are skewed to the right and leptokurtic (that is, exhibit a sharper peak and longer, fatter tails) compared with those in developed open economies. Second, our calibrations show that our model does a very good job approximating these moments.

Under the assumption that international lenders are risk neutral and have rational expectations (and thus fully trust their model), the equilibrium bond prices are simply given by the discounted conditional probability of not defaulting next period.[8] Consequently, the pricing rule in these environments prescribes a strong connection between equilibrium prices

[6]The issue of the scarcity of reliable official data has become more severe in recent years in some emerging economies, such as Argentina, where the government's intervention in the computation of the consumer price index is known worldwide.

[7]Boz, Daude, and Durdu (2011) documents the availability of significantly shorter time series for most relevant economic indicators in emerging economies than in developed ones. For example, in the database from the International Financial Statistics of the IMF, the median length of available GDP time series at a quarterly frequency is 96 quarters in emerging economies, while in developed economies it is 164.

[8]See Arellano (2008) and Aguiar and Gopinath (2006), for example.

and default probability. When calibrated to the data, matching the default frequency to historical levels (the consensus number for Argentina is around 3 percent annually) delivers spreads that are too low relative to those observed in the data.[9] Our methodology breaks this strong connection by introducing a different probability measure, the one in which lenders' uncertainty aversion is manifested. In our framework, there is a strong connection between equilibrium prices and the default probability under this new probability measure. Finally, some recent papers assume instead an ad hoc functional form for the market stochastic discount factor in order to generate sizable bond spreads as observed in the data. Our paper can be seen as providing microfoundations for such functional forms.[10]

From an asset pricing perspective, the key element in generating high spreads while matching the default frequency is a sufficiently negative correlation of the market stochastic discount factor with the country's default decisions. With fears about model misspecification, the stochastic discount factor has an additional component given by the probability distortion inherited in the worst-case density for the borrower's endowment. This probability distortion, which is low when the borrower repays and high when the borrower defaults, induces in general a negative co-movement between the stochastic discount factor and the default decisions of the borrower, which, as mentioned before, is necessary to explain high bond spreads.

In our model with a defaultable asset, this endogenous probability distortion is discontinuous in the realization of the borrower's next-period endowment as a result of the discontinuity in the payoff of the risky bond due to the default contingency. This yields an endogenous hump of the worst-case density over the interval of endowment realization in which default is optimal. This special feature is unique to this current setting. A direct implication of this is that the subjective probability assigns a significantly higher probability to the default event than the actual one. Since we can view the default event as a "disaster event" from the lenders' perspective, this result links to the growing literature on "rare events"; see, for example, Barro (2006). Fears about model misspecification then amplify its effect on both allocations and equilibrium prices, as they increase the lenders' perceived likelihood of these rare events occurring.

[9] Arellano (2008), Borri and Verdelhan (2010), Lizarazo (2010), and Hatchondo, Martínez, and Padilla (2010) use a default frequency of 3 percent per year. Yue (2010) and Mendoza and Yue (2010) target an annual default frequency of 2.78 percent. Also, Reinhart, Rogoff, and Savastano (2003) finds that emerging economies with at least one episode of external default or debt restructuring defaulted roughly speaking three times every 100 years over the period from 1824 to 1999.

[10] See Arellano (2008), Arellano and Ramanarayanan (2012), and Hatchondo, Martínez, and Padilla (2010) for cases using an ad hoc functional form for the market stochastic discount factor.

In this paper we also present two methodological contributions that are of independent interest. The first methodological contribution of this paper relates to the way we solve the model numerically using the discrete state space (DSS) technique, in the context of model uncertainty. Since default is a discrete choice, it can occur that—under the DSS technique—the debt policy rule is not continuous in the current-state variables and prices. In turn, the discontinuity in the debt policy function with respect to bond prices translates into discontinuity in the lenders' Euler equation, which may lead to convergence problems. We handle this technical complication by introducing an i.i.d. preference shock. This preference shock enters additively in the autarky utility value of the borrower's utility when it evaluates the default decision, and it is drawn from a logistic distribution, following McFadden (1981) and Rust (1994). As a result, the default decision, which was originally a discrete variable taking values of 0 or 1, becomes a continuous variable representing a probability that depends on the spread of the borrower's continuation values of repaying and defaulting on the outstanding debt. We show that, as the distribution of the preference shock converges to a point mass at zero (that is, its variance converges to zero), if the equilibrium in the economy with the preference shock converges, it does so to the equilibrium in the economy without preference shock. It therefore implies that for sufficiently small preference shocks, the economy with the preference shock is closed to the original economy.

The second methodological contribution consists of an alternative way of interpreting the detection error probability (DEP) presented in Anderson, Hansen, and Sargent (2003), among others.[11] Roughly speaking, DEP is akin to the type I error, which measures the probability of mistakenly rejecting the true model. Our interpretation links the parameter driving the concerns of model misspecification with the minimum number of observations needed to separate the distorted model from the approximating one, with a certain degree of confidence that is reflected in a chosen probability level.

Related Literature. This paper builds and contributes to two main strands of the literature: sovereign default, and robust control theory and ambiguity aversion or Knightian uncertainty, in particular as applied to asset pricing.

Arellano (2008) and Aguiar and Gopinath (2006) were the first to extend Eaton and Gersovitz's (1981) general equilibrium framework with endogenous default and risk neutral lenders to study the business cycles of emerging economies. Lizarazo (2010) endows the

[11]DEP measures the discrepancy between the approximating and the distorted models. Lenders in our economy are assumed to be concerned about models for the borrower's endowment that are difficult to distinguish from one another given the available dataset.

lenders with constant relative risk aversion (CRRA) preferences. Borri and Verdelhan (2010) have studied the setup with a positive correlation between lenders' consumption and output in the emerging economy in addition to time-varying risk aversion on the lenders' side as a result of habit formation.

From a technical perspective, Chatterjee and Eyingungor (2012) proposes an alternative approach to handle convergence issues. The authors consider an i.i.d. output shock drawn from a continuous distribution with a very small variance. Once this i.i.d. shock is incorporated, they are able to show the existence of a unique equilibrium price function for long-term debt with the property that the return on debt is increasing in the amount borrowed.

To our knowledge, the paper that is the closest to ours is the independent work by Costa (2009). That paper also assumes that foreign lenders want to guard themselves against specification errors in the stochastic process for the endowment of the borrower, but this is achieved in a different form. In our model, lenders are endowed with Hansen and Sargent (2005) multiplier preferences. With these preferences, lenders contemplate a set of alternative models and want to guard themselves against the model that minimizes their lifetime utility. In contrast, in Costa (2009) the worst-case density minimizes the expected value of the bond. Moreover, in Costa (2009) lenders are assumed to live for one period only.

Other recent studies that have focused on business cycles in emerging economies in the presence of fears about model misspecification are Young (2012) and Luo, Nie, and Young (2012). Young (2012) studies optimal tax policies to deal with sudden stops when households and/or agents distrust the stochastic process for tradable total factor productivity shocks, trend productivity, and the interest rate. Luo, Nie, and Young (2012) explores the role of robustness and information-processing constraints (rational inattention) in the joint dynamics of consumption, current account, and output in small open economies.

Finally, our paper relates to the growing literature analyzing the asset-pricing implications of ambiguity. Barillas, Hansen, and Sargent (2009) finds that introducing concerns about robustness to model misspecification can yield combinations of the market price of risk and the risk-free rate that approach Hansen and Jagannathan (1991) bounds. Using a dynamic portfolio choice problem of a robust investor, Maenhout (2004) can explain high levels of the equity premium, as observed in the data. Drechsler (2012) replicates several salient features of the joint dynamics of equity returns, equity index option prices, the risk-free rate, and conditional variances, in the context of Knightian uncertainty. Hansen and Sargent (2010) generates time-varying risk premia in the context of model uncertainty with hidden Markov states. Ju and Miao (2012) considers a consumption-based asset-pricing model with hidden Markov regime-switching processes for consumption and dividends. A representative agent in this pure-exchange economy exhibits generalized recursive smooth preferences, closely related to Klibanoff, Marinacci, and Mukerji's (2005) model of preferences. That model can explain many asset-pricing puzzles. Epstein and Schneider (2008) studies the impact of uncertain information quality on asset prices in a model of learning with investors endowed with recursive multiple-priors utility, axiomatized in Epstein and Schneider (2003).[12]

Roadmap. The paper is organized as follows. Section 2 describes the model. In Section 3 we describe the implications of model uncertainty on equilibrium prices. In Section 4 we calibrate our model to Argentinean data and describe the computational algorithm. In Section 5 we present our quantitative results. Section 6 explains how to interpret the parameter that measures the lenders' preference for robustness to model misspecification. Finally, Section 7 concludes.

2 The Model

In our model an emerging economy interacts with a continuum of identical foreign lenders of measure 1. The emerging economy is populated by a representative, risk-averse household and a government.

The government in the emerging economy can trade a one-period discount bond with atomistic foreign lenders to smooth consumption and allocate it optimally over time. Throughout the paper we refer to the emerging economy as the *borrower*. Debt contracts cannot be enforced and the borrower may decide to default at any point of time. In case the government defaults on its debt, it incurs two types of costs. First, it is temporarily excluded from financial markets. Second, it suffers a direct output loss.

While the borrower fully trusts the probability model governing the evolution of its endowment, which we refer to as the *approximating model*, the lender suspects it is misspecified. From here on, we will use the terms *probability model* and *density*, interchangeably. For this reason, the lender contemplates a set of alternative models that are statistical perturbations of the approximating model, and wishes to design a decision rule that performs well across this set of densities.[13]

[12] More asset-pricing applications with different formulations of ambiguity aversion are Epstein and Wang (1994), Chen and Epstein (2002), Hansen (2007), and Bidder and Smith (2011).

[13] In order to depart as little as possible from Eaton and Gersovitz's (1981) framework, throughout the

Time is discrete $t = 0, 1, \ldots$. For each t, let Y_t be the borrower's exogenous stochastic endowment, where Y_t is the projection of the t-th coordinate of $Y^\infty = (Y_0, Y_1, ..., Y_t, ...)$, where $Y^\infty \in \mathbb{Y}^\infty$ and $\mathbb{Y} \subseteq \mathbb{R}_+$. We use $\mathcal{Y}^t$ to denote the σ-algebra generated by the partial history $Y^t \equiv (Y_0, Y_1, ..., Y_t)$. We assume Y^∞ is a stationary Markov process of order 1, with conditional pdf (with respect to the Lebesgue measure) given by $(y', y) \mapsto f(y'|y)$. Throughout the paper we use Y to denote the random variable and y to denote a particular realization. We use y^L to denote the endowment of the lender, which is chosen to be non-random and constant over time for simplicity.[14]

We follow Arellano (2008) and adopt a recursive formulation for both the borrower's and the lender's problem. We still use t and $t+1$ to denote the current and next period's variables, respectively.

2.1 Timing Protocol

We assume that all economic agents, lenders, and the government (which cares about the consumption of the representative household) act sequentially, choosing their allocations period by period.

The economy can be in one of two stages at the beginning of each period t: financial autarky or with access to financial markets.

The timing protocol within each period is as follows. First, the endowments are realized. If the government has access to financial markets, it decides whether to repay its outstanding debt obligations or not. If it decides to repay, it chooses new bond holdings and how much to consume. Then, atomistic foreign lenders—taking prices as given—choose how much to save and how much to consume. The minimizing agent, who is a metaphor for the lenders' fears about model misspecification, chooses the probability distortions to minimize the lenders' expected utility. Due to the zero-sumness of the game between the lender and its minimizing agent, different timing protocols of their moves yield the same solution. If the government decides to default, it switches to autarky for a random number of periods. While the government is excluded from financial markets, it has no decision to make and simply awaits re-entry to financial markets.

paper we assume that the lender distrusts only the probability model dictating the evolution of the endowment of the borrower, not the distribution of any other source of uncertainty, such as the random variable that indicates whether the borrower re-enters financial markets or not. At the same time, and for the same reason, we assume the extreme case of no doubts about model misspecification on the borrower's side.

[14]Throughout the paper, we assume linear per-period utility for lenders. Under this assumption, considering a fully-trusted stochastic process for the lenders' endowment would not alter equilibrium prices nor the borrower's decisions; see Subsection 3.3 for a discussion.

2.2 Sovereign Debt Markets

Financial markets are incomplete. Only a noncontingent, one-period bond can be traded between the borrower and the lenders. The borrower, however, can default on this bond at any time, thereby adding some degree of state contingency.

Bond holdings of the government and the individual lenders, which are $\mathcal{Y}^{t-1}$-measurable, are denoted by $B_t \in \mathbb{B} \subseteq \mathbb{R}$ and $b_t \in \mathbb{B} \subseteq \mathbb{R}$, respectively, with $\mathbb{B}$ bounded. If it purchases bonds, the government is saving, and bond holdings B_t is positive; otherwise, if it sells bonds, the government borrows from the lenders and B_t is negative. The set $\mathbb{B}$ includes possible borrowing or savings limits.

The borrower can buy a quantity B_{t+1} of bonds at a price q_t. A debt contract is given by a vector (B_{t+1}, q_t) of quantities of bonds and corresponding bond prices. The price q_t depends on the borrower's demand for debt B_{t+1} at time t, and on his endowment y_t, since these variables affect his incentives to default. In this class of models, generally, the higher the level of indebtness and/or the lower the (persistent) borrower's endowment, the greater the chances the borrower will default next period and, hence, the lower the bond prices in the current period.

For each $y \in \mathbb{Y}$, we refer to $q(y, \cdot) : \mathbb{B} \to \mathbb{R}_+$ as the bond price function, for y. Thus, we can define the set of debt contracts available to the borrower for a given y as the graph of $q(y, \cdot)$.[15]

2.3 Borrower's Preferences

A representative household in the emerging economy derives utility from consumption of a single good in the economy. Its preferences over consumption plans can be described by the expected lifetime utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

where E_0 denotes the mathematical expectation operator under f conditional on time zero information, $\beta \in (0, 1)$ denotes the time discount factor, and the period utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and strictly concave, and satisfies the Inada conditions.

Note that the assumption that the representative household and the government fully trust the approximating model f is embedded in E_0.

The government in this economy, which is benevolent and maximizes the household's

[15] The graph of a function, $f : \mathbb{X} \to \mathbb{Y}$, is the set of $\{(x, y) \in \mathbb{X} \times \mathbb{Y} \colon y = f(x) \textit{ and } x \in \mathbb{X}\}$.

utility (1), may have access to international financial markets, where it can trade a one-period discount bond with the foreign lenders. While the government has access to the financial markets, it can sell or purchase bonds from the lenders and make a lump-sum transfer across households to help them smooth consumption over time. Debt is also used to front-load consumption, as the borrower is more impatient than the international lenders, that is, $\beta < \gamma$ (where γ is the discount factor for the representative lender).

2.4 Borrower's Problem

For each (y_t, B_t), let $V(y_t, B_t)$ be the value (in terms of lifetime utility) for the borrower of having the option to default, given an endowment of y_t and outstanding bond holdings equal to B_t. Formally, the borrower's value of having access to financial markets $V(y_t, B_t)$ is given by

$$V(y_t, B_t) = \max \{V_A(y_t), V_R(y_t, B_t)\},$$

where $V_A(y_t)$ is the value of exercising the option to default, given an endowment of y_t, and $V_R(y_t, B_t)$ is the value of repaying the outstanding debt, given state (y_t, B_t). Throughout the paper we use subscripts A and R to denote the values for *autarky* (or *default*) and *repayment*, respectively.

Every period the government enters with access to financial markets, it evaluates the present lifetime utility of households if debt contracts are honored against the present lifetime utility of households if they are repudiated. If the former outweighs the latter, the government decides to comply with the contracts, pays back the debt carried from the last period B_t, and chooses next period's bond holdings B_{t+1}. Otherwise, if the utility of defaulting on the outstanding debt and switching to financial autarky is higher, the government decides to default on the sovereign debt.

Consequently, the value of repayment $V_R(y_t, B_t)$ is

$$\begin{aligned} V_R(y_t, B_t) = \max_{B_{t+1} \in \mathbb{B}} u(c_t) + \beta \int_{\mathbb{Y}} V(y_{t+1}, B_{t+1}) f(y_{t+1}|y_t) dy_{t+1} \\ s.t.\ c_t = y_t - q_t B_{t+1} + B_t, \end{aligned}$$

where $q_t \equiv q(y_t, B_{t+1})$.[16]

[16]We also assume throughout the paper that the borrower faces an exogenous borrowing constraint, $B_{t+1} \geq \underline{B}$, which rules out Ponzi schemes and is assumed never to be binding; such a constraint is embedded in $\mathbb{B}$.

Finally, the value of autarky $V_A(y_t)$ is

$$V_A(y_t) = u(y_t - \phi(y_t)) + \beta \int_{\mathbb{Y}} \left((1-\pi)V_A(y_{t+1}) + \pi V(y_{t+1}, 0)\right) f(y_{t+1}|y_t) dy_{t+1},$$

where π is the probability of re-entering financial markets next period.[17] In that event, the borrower enters next period carrying no debt, $B_{t+1} = 0$.[18] The function $\phi : \mathbb{Y} \to \mathbb{Y}$ such that $y \geq \phi(y) \quad \forall y \in \mathbb{Y}$ represents the ad hoc direct output cost, in terms of consumption units, that the borrower incurs when excluded from financial markets. This output loss function is consistent with evidence that shows that countries experience a fall in output in times of default due to the lack of short-term trade credit.[19] Notice that in autarky the borrower has no decision to make and simply consumes $y_t - \phi(y_t)$.

The default decisions can be characterized in terms of default sets and a default indicator. Let $D : \mathbb{B} \to 2^{\mathbb{Y}}$. Then we define the default set for a given debt level B

$$D(B) \equiv \{y : V_R(y, B) < V_A(y)\}$$

to be the set of endowment realizations for which the government finds it optimal to default.

In a similar fashion, let $\delta : \mathbb{Y} \times \mathbb{B} \to \{0, 1\}$ denote the default indicator, which takes value 0 in case of default; and 1, otherwise; that is,[20]

$$\delta(y, B) = \mathcal{I}\left\{V_R(y, B) \geq V_A(y)\right\}.$$

2.5 Lenders' Preferences and their Fears about Model Misspecification

We assume that the lenders have per-period payoff linear in consumption, while also being uncertainty averse or ambiguity averse. The reason for this is that we want to highlight the

[17] As in Arellano (2008), we do not model the exclusion from financial markets as an endogenous decision by the lenders. By modeling this punishment explicitly in long-term financial relationships, Kletzer and Wright (1993) shows how international borrowing can be sustained in equilibrium through this single credible threat.

[18] Notice that we assume there is no debt renegotiation nor any form of debt restructuring mechanism. Yue (2010) models a debt renegotiation process as a Nash bargaining game played by the borrower and lenders. For more examples of debt renegotiation, see Benjamin and Wright (2009) and Pitchford and Wright (2012). Pouzo (2010) assumes a debt restructuring mechanism in which the borrower receives random exogenous offers to repay a fraction of the defaulted debt. A positive rate of debt recovery gives rise to positive prices for defaulted debt that can be traded among lenders in secondary markets.

[19] Mendoza and Yue (2010) endogenizes this output loss as an outcome that results from the substitution of imported inputs by less-efficient domestic ones as credit lines are cut when the country declares a default.

[20] For any set, A, the function $\mathcal{I}\{A\}$ is 1 if $x \in A$ and 0 otherwise.

effects of uncertainty aversion on the prices, and other equilibrium quantities, in an otherwise standard dynamic general equilibrium model.

The lenders distrust the probability model, which dictates the evolution of the endowment of the borrower, given by the *approximating model* f. For this reason, they contemplate a set of alternative densities that are statistical perturbations of the approximating model, and they wish to design a decision rule that performs well over this set of priors. These alternative conditional densities, denoted by $\widetilde{f}$, are assumed to be absolutely continuous with respect to f; that is, for all $A \in \mathcal{Y}$ and $y^t \in \mathbb{Y}^t$, if $\int_A f(y_{t+1}|y_t)dy_{t+1} = 0$, then $\int_A \widetilde{f}_t(y_{t+1}|y^t)dy_{t+1} = 0$.[21]

To construct any of these distorted probabilities $\widetilde{f}_t$ we can use a non-negative $\mathcal{Y}^t$-measurable function m_{t+1}, that takes the form of a conditional likelihood ratio, that is,

$$m_{t+1}(y_{t+1}|y^t) = \begin{cases} \frac{\widetilde{f}_t(y_{t+1}|y^t)}{f(y_{t+1}|y_t)} & \text{if } f(y_{t+1}|y_t) > 0 \\ 1 & \text{if } f(y_{t+1}|y_t) = 0 \end{cases}.$$

Note that $\int m_{t+1}(y_{t+1}|y^t)f(y_{t+1}|y_t)dy_{t+1} = 1$. Henceforth, let $\mathcal{M} \equiv \{g\colon \mathbb{Y} \to \mathbb{R}_+ : \int g(y_{t+1})f(y_{t+1}|y_t)dy_{t+1} = 1\}$.

For any given history y^t, the discrepancy between the distorted and approximating probability distributions, $\widetilde{f}_t(\cdot \mid y^t)$ and $f(\cdot|y_t)$, respectively, is measured by the relative entropy, which takes the form

$$\mathcal{E}(m_{t+1}(\cdot|y^t)) \equiv \int m_{t+1}(y_{t+1}|y^t) \log m_{t+1}(y_{t+1}|y^t) f(y_{t+1}|y_t)dy_{t+1}.$$

Following Hansen and Sargent (2008) and references therein, to express fears about model misspecification we endow lenders with multiplier preferences. We can think of the lenders as playing a zero-sum game against their fictitious minimizing agent, who represents their doubts about model misspecification. While the lenders choose bond holdings to maximize their utility, the minimizing agent chooses distorted densities $\widetilde{f}_{t+1}$, or equivalently conditional likelihood ratios m_{t+1}, to minimize it. Thus, preferences over consumption plans for lenders are then described as follows. For any given consumption plan $\mathbf{c}^L$ the lifetime utility over such a plan is given by[22]

$$U_0(\mathbf{c}^L; y_0) \equiv \min_{(m_{t+1})_t} \sum_{t=0}^{\infty} \gamma^t \int_{\mathbb{Y}^t} M_t\{c_t^L + \theta\gamma\mathcal{E}(m_{t+1})\}P(dy^t), \tag{2}$$

[21]Note that the distorted densities $\widetilde{f}$ do not necessarily inherit the properties of f, such as its Markov structure.

[22]A consumption plan is a sequence $(c_t^L)_t$ where c_t^L is $\mathcal{Y}^t$-measurable.

where $\gamma \in (0,1)$ is the discount factor, the parameter $\theta \in (\underline{\theta}, +\infty]$ is a penalty parameter that measures the degree of concern about model misspecification, and $M_{t+1} = m_{t+1}M_t = \prod_{\tau=1}^{t+1} m_\tau M_0$ with $M_0 = 1$ and $m_{t+1}(\cdot|y_t) \in \mathcal{M}$.

The non-negative random variable M_t is a likelihood ratio, that is, $M_t = \frac{\widetilde{f}_t(y^t)}{f_t(y^t)}$, which induces the aforementioned change of measure.[23] The minimization problem conveys the ambiguity aversion. Through the entropy term, the minimizing agent is penalized whenever she chooses distorted probabilities that differ from the approximating model. The higher the value of θ, the more the minimizing agent is penalized. In the extreme case of $\theta = +\infty$, there are no concerns about model misspecification and we are back to the standard environment where both borrower and lenders share the same model, given by f.

Following the results in Appendix D, we can cast the previous equation (2) recursively for any (t, y^t) as

$$U_t(\mathbf{c}^L; y^t) = \min_{m_{t+1}\in\mathcal{M}} \left\{ c_t^L + \theta\gamma\mathcal{E}(m_{t+1}) + \gamma \int_{\mathbb{Y}} m_{t+1} \left(U_{t+1}(\mathbf{c}^L; y^t, y_{t+1}) \right) f(y_{t+1}|y_t) dy_{t+1} \right\}, \quad (3)$$

where $U_t(\mathbf{c}^L; y^t)$ is the lifetime utility of following the continuation of the plan $\mathbf{c}^L$ from (t, y^t) onwards.

It is easy to see that the minimization problem yields the following specification for m_{t+1}

$$m^*_{t+1}(y_{t+1}|y^t) = \frac{\exp\left\{ -\frac{U_{t+1}(\mathbf{c}^L; y^t, y_{t+1})}{\theta} \right\}}{\int_{\mathbb{Y}} \exp\left\{ -\frac{U_{t+1}(\mathbf{c}^L; y^t, y_{t+1})}{\theta} \right\} f(y_{t+1}|y_t) dy_{t+1}}. \quad (4)$$

Through her choice of m^*_{t+1}, the minimizing agent pessimistically twists the conditional distribution f by putting more weight on continuation outcomes associated with lower utility for the lenders.

2.6 Lenders' Problem

In our particular environment, if the optimal consumption plan is chosen, subject to the budget constraint, taking as given prices and actions by the government in the recursive equilibrium, $U_t(\mathbf{c}^L; y^t)$ in the previous section becomes $W_R(y_t, B_t, b_t)$ or $W_A(y_t)$; where $W_R(y_t, B_t, b_t)$ is the equilibrium value (in lifetime utility) for the representative lender when there is access to financial markets, given the state of the economy (y_t, B_t, b_t), and $W_A(y_t)$ is analogously defined, but when the borrowing economy has no access to financial markets.

[23] Also, M_t is martingale and its existence is guaranteed by the Radon-Nikodym theorem.

Since lenders are atomistic, each individual lender takes as given the aggregate debt. The lender has a *perceived* law of motion for this variable, which only in equilibrium will be required to coincide with the actual one. We denote b_t as the individual lender's debt, while B_t refers to the representative lender's debt.

When lender and borrower can engage in a new financial relationship, the lender's min-max problem, at state (y_t, B_t, b_t), is given by:

$$W_R(y_t, B_t, b_t) = \min_{m_R \in \mathcal{M}} \max_{c_t^L, b_{t+1}} \left\{ c_t^L + \theta\gamma\mathcal{E}(m) + \gamma \int_{\mathbb{Y}} m(y_{t+1}) \left(W(y_{t+1}, B_{t+1}, b_{t+1}) \right) f(y_{t+1}|y_t) dy_{t+1} \right\} \tag{5}$$

$$s.t. \; c_t^L = y^L + q_t b_{t+1} - b_t \tag{6}$$

$$B_{t+1} = \Gamma(y_t, B_t), \tag{7}$$

where $W(y_{t+1}, B_{t+1}, b_{t+1}) \equiv \delta(y_{t+1}, B_{t+1}) W_R(y_{t+1}, B_{t+1}, b_{t+1}) + (1 - \delta(y_{t+1}, B_{t+1})) W_A(y_{t+1})$ is the value for the lender when the borrower is given the option to default, and $\Gamma : \mathbb{Y} \times \mathbb{B} \to \mathbb{B}$ is the perceived law of motion of the individual lender for the debt holdings of the borrower, B_{t+1}. Note that the optimal choice, m_R, is a mapping from $(y_t, B_{t+1}, b_{t+1}) \in \mathbb{Y} \times \mathbb{B}^2$ to $\mathcal{M} \subseteq L^1(f_Y)$.

A few remarks are in order regarding equation (5). First, lenders receive every period a nonstochastic endowment given by y^L. Since the per-period utility is linear in consumption, the level of y^L does not affect the equilibrium bond prices, bond holdings, and default strategies in our original economy (see Subsection 3.3). Moreover, allowing for a stochastic endowment for lenders, insofar as every realization is sufficiently large and there are no doubts about its stochastic process, should affect neither the equilibrium borrower's allocations nor prices. Second, lenders can only trade one-period risky bonds. We could allow the lenders to trade a zero net supply riskless claim to one unit of consumption next period. Since all lenders are identical, no trade in such a claim takes place in equilibrium. Introducing this riskless claim is, however, useful to determine a risk-free rate r_t^f and thereby compute bond spreads. It turns out that we could relax this assumption by letting lenders have access to saving-borrowing technology at a nonstochastic gross risk-free rate given by $1 + r^f = 1/\gamma$. By doing so, we would not be altering the equilibrium bond prices, bond holdings, or default strategies in our original economy, as shown later in Subsection 3.3.

In financial autarky, as with the borrower, the lender has no decision to make. The lender's min-max problem, at state (y_t), is thus given by

$$W_A(y_t) = \min_{m_A \in \mathcal{M}} \left\{ y^L + \theta\gamma\mathcal{E}(m) + \gamma \int_{\mathbb{Y}} m(y_{t+1}) \left((1-\varrho)W_A(y_{t+1}) + \varrho W(y_{t+1},0,0)\right) f(y_{t+1}|y_t) dy_{t+1} \right\},$$

where ϱ is a random variable indicating reentry to financial markets, that is equal to 0 with probability $(1-\pi)$ and 1 with probability π. Note that the optimal choice, m_A, is a mapping from $\mathbb{Y}$ to $\mathcal{M} \subseteq L^1(f_Y)$.

In contrast with the borrower's case, no output loss is assumed for the lender during financial autarky.

2.7 Recursive Equilibrium

We are interested in a Markov recursive equilibrium in which all agents choose sequentially.

Definition 2.1. *A collection of policy functions* $\{c, c^L, B, b, m, \delta\}$ *is given by mappings for consumption* $c : \mathbb{Y} \times \mathbb{B} \to \mathbb{R}_+$ *and* $c^L : \mathbb{Y} \times \mathbb{B}^2 \to \mathbb{R}_+$*, bond holdings* $B : \mathbb{Y} \times \mathbb{B} \to \mathbb{B}$ *and* $b : \mathbb{Y} \times \mathbb{B}^2 \to \mathbb{B}$ *for borrower and individual lender, respectively; and, probability distortions* $m_R : \mathbb{Y} \times \mathbb{B}^2 \to L^1(f_Y)$*,* $m_A : \mathbb{Y} \to L^1(f_Y)$ *and default decisions,* $\delta : \mathbb{Y} \times \mathbb{B} \to \{0, 1\}$*.*

Definition 2.2. *A collection of value functions* $\{V_R, V_A, W_R, W_A\}$ *is given by mappings* $V_R : \mathbb{Y} \times \mathbb{B} \to \mathbb{R}$*,* $V_A : \mathbb{Y} \to \mathbb{R}$*,* $W_R : \mathbb{Y} \times \mathbb{B}^2 \to \mathbb{R}$*,* $W_A : \mathbb{Y} \to \mathbb{R}$*.*

Definition 2.3. *A price schedule is given by* $q : \mathbb{Y} \times \mathbb{B} \to \mathbb{R}_+$*.*

Definition 2.4. *A recursive equilibrium is a collection of policy functions* $\{c^*, c^{L,*}, B^*, b^*, m_R^*, m_A^*, \delta^*\}$*, a collection of value functions* $\{V_R^*, V_A^*, W_R^*, W_A^*\}$*, a perceived law of motion for the borrower's bond holdings, and a price schedule such that:*

1. *Given perceived laws of motion for the debt and price schedule, policy functions, probability distortions, and value functions solve the borrower and individual lender's min-max problems.*

2. *Bond prices* $q(y, B^*(y, B))$ *clear the financial markets, that is,*

$$B^*(y, B) = b^*(y, B, B), \ \forall (y, B) \in \mathbb{Y} \times \mathbb{B}.$$

3. *The actual and perceived laws of motion for debt holdings coincide, that is,* $B^*(y, B) = \Gamma(y, B)$*.*

After imposing the market clearing condition given by point 3 above, vector (y_t, B_t) is sufficient to describe the state variables for any agent in this economy. Hence, from here on, we consider (y_t, B_t) as the state vector, common to the borrower and the individual lenders.

3 Equilibrium Bond Prices and Probability Distortions

3.1 Euler Equation and Probability Distortions

In equilibrium, for each state of the economy (y_t, B_t) only one debt contract is traded between the borrower and the lenders and, hence, we observe a particular quantity of new bond holdings, $B^*(y_t, B_t)$, with an associated price q_t.

By taking FOCs with respect to b_{t+1} in the lender's optimization problem, and imposing the equilibrium conditions, we derive the lenders' intertemporal Euler equation for the state (y_t, B_t)

$$q_t = \gamma \int_{D(B^*_{t+1})^C} m^*_R(y_{t+1}, y_t, B^*_{t+1}, B^*_{t+1}) f(y_{t+1}|y_t) dy_{t+1}, \tag{8}$$

where $B^*_{t+1} \equiv B^*(y_t, B_t)$ and $m^*_R(\cdot; y_t, B^*_{t+1}, B^*_{t+1})$ is the optimal multiplicative distortion to the conditional density f defined analogously to expression (4), when the borrower repays in the current state (y_t, B_t). Henceforth, in order to ease the notational burden, we will use $m^*_R(y_{t+1}, y_t, B^*_{t+1})$ to denote $m^*_R(y_{t+1}, y_t, B^*_{t+1}, B^*_{t+1})$.

$$m^*_R(y_{t+1}, y_t, B^*_{t+1}) = \frac{\exp\left\{-\frac{W(y_{t+1}, B^*_{t+1}, B^*_{t+1})}{\theta}\right\}}{\int_{\mathbb{Y}} \exp\left\{-\frac{W(y_{t+1}, B^*_{t+1}, B^*_{t+1})}{\theta}\right\} f(y_{t+1}|y_t) dy_{t+1}}. \tag{9}$$

Note also that for any value of $\theta \in (\underline{\theta}, +\infty]$ —with and without misspecification concerns— the gross risk-free rate $1 + r^f$ is equal to the reciprocal of the lenders' discount factor γ.

Let us analyze first the case without model misspecification. In the absence of fears about model uncertainty, that is, $\theta = +\infty$, the probability distortion vanishes, that is, $m^*_R(\cdot, y_t, B^*_{t+1}) = 1$, and equilibrium bond prices then turn out to be the discounted probability, computed under f, of not defaulting next period, as in Arellano (2008).

Under model uncertainty, the lender in this economy distrusts the conditional density f and wants to guard himself against a worst-case distorted density for y_{t+1}, given by $m^*_R(\cdot, y_t, B^*_{t+1}) f(\cdot|y_t)$. The fictitious minimizing agent, who represents its doubts about model misspecification, will be selecting this worst-case distorted density by slanting probabilities toward the states associated with low continuation utility for the lender, as observed from the twisting formula given by equation (9). In the presence of default risk, the states associated with low utility coincide with the states in which the borrower defaults and therefore the lender receives no repayment.

Figure 1 illustrates the optimal distorting of the probability of next period realization of endowment y_{t+1}, given current state (y_t, B_t) with access to financial markets.[24] In this figure we plot the conditional approximating density and the distorted density for y_{t+1}, as well as its corresponding probability distortion m_R^*. The shaded area corresponds to the range of values for the realization of y_{t+1} in which the borrower defaults.

In order to minimize lenders' expected utility, the minimizing agent places a discontinuous probability distortion $m_R^*(\cdot)$ over next period realizations of y_{t+1}, with values strictly larger than 1 over the default interval, and strictly smaller than 1 where repayment is optimal.[25] By doing so, the minimizing agent takes away probability mass from those states in which the borrower does not default, and puts it in turn on those low realizations of y_{t+1} in which default is optimal for the borrower. The discontinuity of $m_R^*(\cdot)$ follows from the discontinuity of the lenders' utility value with respect to y_{t+1}, which in turn is due to the discontinuity in next-period payoff of the bond as function of y_{t+1}. Discrete jumps in the payoff structure are therefore key to generating a discontinuous stochastic discount factor in our environment. For the particular state vector (y_t, B_t) in consideration, the conditional default probability under the approximating model is 2.6 percent quarterly, while under the distorted one it is 5.7 percent, more than twice as high.

The tilting of the probabilities by the minimizing agent generates an endogenous hump of the distorted density over the interval of y_{t+1} associated with default risk, as observed in Figure 1. The bi-peaked form of the resulting distorted conditional density is nonstandard in the robust control literature, in which it typically displays only a shift in the conditional mean from the approximating one.[26] Table 5 in Appendix A reports distortions in several

[24]A low endowment y_t and high debt level B_t were suitably chosen to have considerable default risk under the approximating density. The current endowment level y_t corresponds to two conditional standard deviations below its unconditional mean, and the bond holdings B_t are set to the 25th percentile of its unconditional distribution in the simulations. B_{t+1} is computed using the optimal debt policy, that is, $B_{t+1} = B^*(y_t, B_t)$.

[25]Note that, even with a zero recovery rate on defaulted debt, m_R^* is not exactly flat over the default interval. To understand why this is the case, note from equation (2) that the lenders' value of repayment depends on future relative entropies $\xi(\cdot)$. Consequently, future probability distortions would also vary with current y_t, since different realizations of it would typically be associated with different amounts of default risk (under the approximating model) in the future, as debt positions and default decisions may differ. A similar argument explains why m_R^* is not flat either over the nondefault interval.

[26]See, for example, Barillas, Hansen, and Sargent (2009) and Anderson, Hansen, and Sargent (2003).

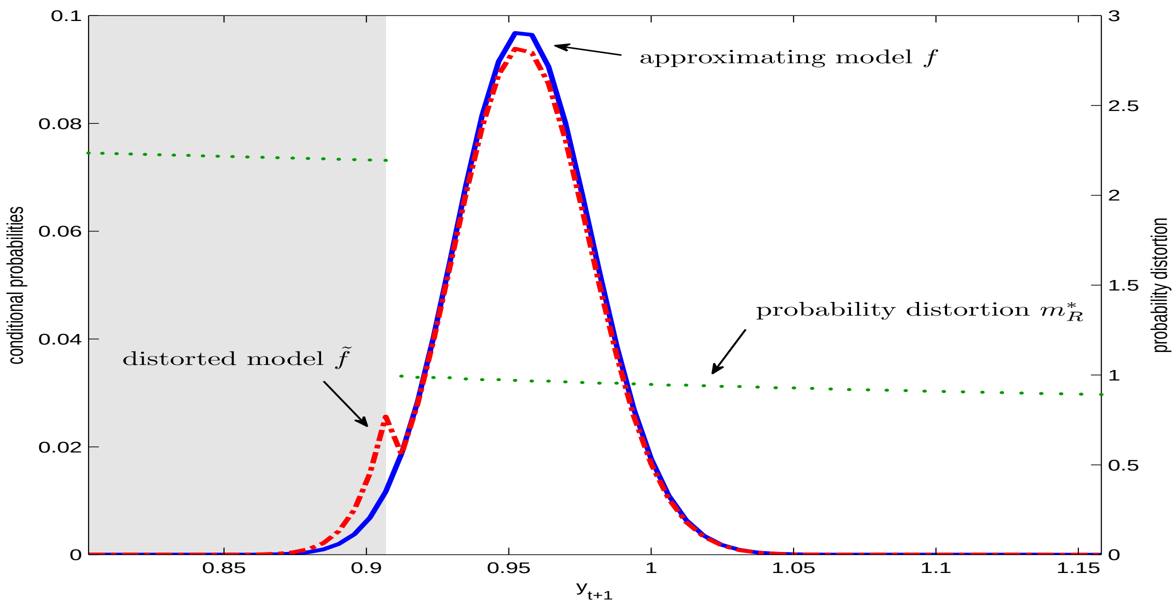

Figure 1: Approximating and distorted densities.

moments of y_t for our economy. The difference arises from the asset payoff structure.

Sovereign default events in our model can be interpreted as "disaster events", which, in our economy, emerge endogenously from the borrower's decision making and the lack of enforceability of debt contracts. Fears about model misspecification in turn amplify their effect on both allocations and equilibrium prices, as they increase the perceived likelihood in the minds of the lenders of these rare events occurring.

Probability distortions are state-dependent. The default risk under the approximating density and the quantity of bonds carried over to next period, which the borrower can default on, affect the extent to which the minimizing agent distorts lenders' beliefs. Figure 2 shows the approximating and distorted density of next period y_{t+1} for different combinations of current endowment and bond holdings, (y_t, B_t).[27]

By comparing the two panels in the top row (or the bottom row), we can see how the probability distortion changes with the level of current debt. In this general equilibrium framework, we need to take into account the optimal debt response of the borrower for the current state of the economy. For the state vectors in consideration, the higher the current level of indebtness B_t, the more debt the borrower optimally chooses to carry into the next

[27]Low and high endowment y_{t+1} correspond to one conditional standard deviation below and above the unconditional mean of output, respectively. Also, low debt B_t is given by the 75th percentile of the debt unconditional distribution, and high debt B_t corresponds to the 25th percentile.

period, B_{t+1}. The *perceived* probability of default next period rises for two reasons as current bond holdings B_t increase. First, the interval of realizations of y_{t+1} for which the borrower defaults is enlarged. The larger the quantity of bonds that the borrower has to repay at $t+1$, the greater the incentives it would have to not do it. Consequently, the default risk under the approximating model is higher. Also, those new states on which there is default with higher debt now become low-utility states for the lender, and hence probability distortions $m^*_{R,t+1}$ larger than 1 are assigned to them in the new, worst-case density. Second, since for these cases, more bond holdings B_{t+1} are carried into the next period, more is at stake for the lender, as the potential losses in the event of default are larger. Hence, the probability mass on the default states would be even higher than before. In other words, the probability distortions over the default interval would be even higher, and the opposite would occur for the ones on the nondefault interval of realizations of y_{t+1}. Notice from Figure 2 that the gap between $m^*_{R,t+1}$ associated with default states and repayment states widens as current debt B_t increases.

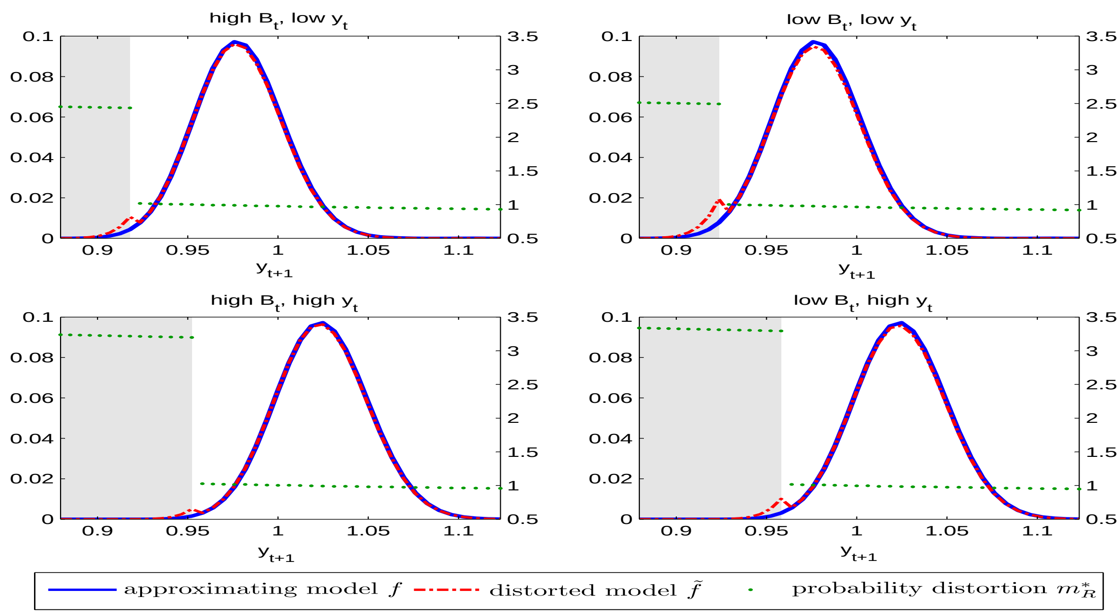

Figure 2: Approximating and distorted densities for different state vectors (y_t, B_t).

By comparing the two panels in the left-side column (or the right-side column) we can see how the probability distortion changes with the level of current endowment. Due to the persistence of the stochastic process for $(y_t)_t$, the lower the current endowment, y_t, the lower is the conditional mean of next period's endowment, y_{t+1} of the approximating density. For the state vectors considered here, the agent chooses relatively more new bond holdings the higher the current endowment y_t. This follows from the interplay of two driving forces: first, consumption smoothing motives, second, output costs of default increasing in the endowment level. The higher the endowment y_{t+1}, the more severely the borrower is punished if it defaults. As the incentives to default are smaller, the returns are lower, or equivalently the bond prices demanded by the lenders are higher, for the same levels of debt. Facing relatively cheaper debt, the borrower responds by borrowing more. In this way, more bond holdings B_{t+1} widens the intervals of y_{t+1}-realizations for which default is optimal. At the same time, probability distortions over the new default interval become relatively larger for similar reasons as discussed previously when debt B_t rises. In these cases, the *perceived* probability distortions, however, decrease due to the rightward shift of the conditional mean of y_{t+1}, as endowment y_t increases. These graphs therefore show how the distorted probability of default is state dependent, in the sense that is relatively higher in bad times—states of low y and high B— than in good times—states of high y and low B.

3.2 Probability Distortions and the Stochastic Discount Factor

As explained in the previous section, the discrepancy between probability models over the range of y_{t+1}-realizations where default is optimal is key to generating high bond spreads in the model while, at the same time, keeping the actual default frequency at historical levels. To see this, we can recast the Euler equation (8) as the standard asset-pricing equation:

$$q_t = E_t(SDF_{t+1} \times \delta^*_{t+1}) = \frac{1}{1+r^f}(1 - P_t(\delta^*_{t+1} = 1)) + Cov_t(SDF_{t+1}, \delta^*_{t+1}),$$

where $SDF_{t+1} \equiv \gamma m^*_R$ stands for the market stochastic discount factor at $t+1$, the default indicator δ_{t+1} represents the payoff of the risky bond, taking a value of 1 when the borrower repays and 0 when default occurs, and P_t denotes the conditional probability computed under the approximating density.

From the equation above it is necessary to have the covariance term sufficiently negative to generate low bond prices in equilibrium, or equivalently, high bond returns. This means that it is necessary that the stochastic discount factor typically be high when the borrower defaults. Lenders have to value consumption more when default events occur.

When $\theta = +\infty$, the fears about model misspecification vanish and we are back to the case with risk-neutral lenders with rational expectations, as in Arellano (2008). Note that the stochastic discount factor is equal to γ, and, therefore, the covariance between the market stochastic discount factor and the borrower's default decisions is zero. This explains why it is not possible to generate high enough spreads in that particular environment.

With uncertainty-averse lenders, the covariance term is no longer zero. The modified stochastic discount factor is then given by two multiplicative components, $\gamma m_R^*(\cdot, y_t, B_{t+1}^*)$. In addition to the discount factor of the lender, we have the probability distortion m_R^* inherited in the worst-case density for the endowment of the borrower. As mentioned earlier, this probability distortion is typically low when the borrower repays and high when the borrower defaults. It therefore induces a desired negative co-movement between the stochastic discount factor and the default decisions of the borrower.[28]

A natural question is whether risk aversion on the lenders' side with time separable preferences could generate a stochastic discount factor, negatively correlated with default decisions δ_{t+1}, that could help account for low bond prices, while preserving the default frequency at historical low levels. We explore this in Section 5 and Appendix C. Our findings indicate that in our calibrated economy with CRRA separable preferences for the lender this is not the case, that is, plausible degrees of risk aversion on the lenders' side are not sufficient to generate high bond returns. See Table 6 for details. These results are also consistent with the findings by Lizarazo (2010) and Borri and Verdelhan (2010).[29]

Even if sufficiently high values of risk aversion could eventually recover the high spreads shown in the data, doing so, however, would lower the risk-free rate to levels far below those exhibited in the data, in line with the risk-free rate puzzle in Weil (1989).[30]

In our environment with model uncertainty, however, the extent to which lenders are uncertainty-averse does not affect the equilibrium gross risk-free rate, given by the reciprocal of γ, as their period utility function is linear in consumption.[31]

[28]See Appendix B for a plot of the correlation between $m_{R,t+1}^*$ and δ_{t+1}.

[29]This is analogous to the equity premium puzzle result studied in Mehra and Prescott (1985).

[30]Note that the stochastic process assumed for y_t is stationary. If we added a positive trend, the risk-free rate would be rising, rather than decreasing, with the increase of the lenders' coefficient of risk aversion.

[31]In our model with linear lenders' per-period utility, equilibrium prices depend exclusively on economic fundamentals of the borrowing economy and the lenders' preference for robustness. It is noteworthy to remark that adding curvature on the per-period utility or fears about misspecification *for the lenders' endowment stochastic process*, will, in general, lead to equilibrium prices that also depend on international lenders' characteristics such as their total wealth and investment flows, or more generally, on global macroeconomic factors, in line with the empirical findings of Longstaff, Pan, Pedersen, and Singleton (2011). This seems to be an interesting extension to pursue in future research.

3.3 Irrelevance of Lenders' Wealth on Equilibrium Prices

In the next two lemmas we present useful equilibrium results for solving and calibrating the model. These results allow us to relax some assumptions on our environment without affecting some key equilibrium objects. The formal statements and their proofs are relegated to Appendix E; here we present an informal statement for the sake of presentation and to ease the notational burden.

The first lemma shows the irrelevance of the relative size of the lenders' nonstochastic endowment to equilibrium bond prices and the borrower's allocations.

LEMMA 3.1. *Consider an arbitrary recursive equilibrium with lenders' nonstochastic endowment given by* y^L. *Then, for any other nonstochastic endowment* $\hat{y}^L \neq y^L$, *there exists a recursive equilibrium with identical bond prices and borrower's allocations.*

The proof (and formal statement of the lemma) is relegated to Appendix E. The intuition of the proof follows from the linearity of the lenders' per-period utility on consumption, and the fact that the probability distortions depend on the absolute value of the spread in the lenders' continuation values across states. In fact, by analogous arguments, it is possible to show that the result in Lemma 3.1 extends to the stochastic process for lenders' endowment, provided that this process is fully trusted by the lender.

The second result refers to the set of financial assets available to the lenders. In addition to the risky bond trading, lenders so far were allowed to trade a riskless claim exclusively among themselves. Since all lenders are identical, no trade of the riskless claim occurs in equilibrium. The next lemma shows that even if lenders are allowed to borrow or save at a given gross risk-free rate in credit markets for developed economies, for example, are able to invest in U.S. treasury bills, this does not affect either the equilibrium bond prices or the borrower's allocation. More precisely, we show that for any recursive equilibrium in this extended economy there is a recursive equilibrium in the original economy without risk-free asset holdings and identical bond prices and borrower's allocations.

LEMMA 3.2. *For any set of recursive equilibria in the economy with nonzero risk-free asset holdings, there exists a recursive equilibrium in the set such that equilibrium prices and the borrower's allocations coincide with the recursive equilibrium without risk-free asset holdings (see definition 2.4).*

The intuition for this proposition is the following (the proof and formal statement of the lemma are relegated to Appendix E). In our original environment, lenders care only about the distorted present value of cash flows from the bond trading (and the associated entropy). Introducing a risk-free asset would not help lenders to smooth that distorted present value across states. Therefore, since, in our setting, a risk-free bond is not useful to hedge against the default risk, any risk-free asset position cannot alter the probability distortions in equilibrium, and hence bond prices.[32]

These two results have important implications for our calibrations. Due to the first result, there is no need to identify who the lenders are in the data, and, in particular, to find a good proxy of their income relatively to the borrower's endowment. Due to the second result, when solving the model numerically, we do not need to keep track of the total wealth of lenders beyond that consisting of risky bonds.

4 Calibration

In this section we analyze the quantitative implications of our model for Argentina. To do so, we specify our choices for functional forms and calibrate some parameter values to match key moments in the data for the Argentinean economy. The period considered spans from the second quarter of 1983 to the last quarter of 2001, when Argentina defaulted on its foreign debt. In Table 1 we present the parameter values for the benchmark model.

For the calibration of our model, we consider the following functional forms. The period utility function for the borrower is assumed to have the CRRA form, that is, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, where σ is the coefficient of relative risk aversion.

We assume that the endowment of the borrower follows a log-normal AR(1) process,

$$\log y_{t+1} = \rho \log y_t + \sigma_\varepsilon \varepsilon_{t+1},$$

where the shock $\varepsilon_{t+1} \sim i.i.d.\mathcal{N}(0,1)$. As shown in Proposition 3.1, the (nonstochastic) lenders' endowment does not affect the equilibrium bond prices and borrower's allocations, allowing us to circumvent the subtle challenge of providing a good proxy for lenders' consumption or income. We therefore set the lenders' log endowment to 1.[33]

Following Chatterjee and Eyingungor (2012), we consider $\phi(y) = \max\{0, \kappa_1 y + \kappa_2 y^2\}$ as the output costs of default, with $\kappa_2 > 0$. As explained later, our calibrated output costs play a key role in generating desired business cycle features for emerging economies, in particular

[32] The fact that the gross risk-free rate is $1/\gamma$ implies that the lenders have no interest in any particular time path for consumption.

[33] In our calibration, with endowment equal to e, lenders' consumption is always positive.

the volatility of bond spreads, in the context of lenders' model uncertainty.

The coefficient of relative risk aversion for the borrower σ is set to 2, which is standard in the sovereign default literature. The re-entry probability π is set to 0.0385, as in Chatterjee and Eyingungor (2012), implying an average period of 6.5 years of financial exclusion and consistent with the estimates by Benjamin and Wright (2009).[34] The lenders' discount factor γ is set equal to the reciprocal of a risk-free rate of 1.7 percent, which is the average quarterly interest rate of a five-year U.S. treasury bond for the period in consideration.

We estimate the parameters ρ and σ_ϵ for the log-normal AR(1) process for the endowment of the borrower, using output data for Argentina. Time series at a quarterly frequency for output, consumption, and net exports for Argentina are taken from the Ministry of Finance (MECON). All these series are seasonally adjusted, in logs, and filtered using a linear trend. Net exports are computed as a percentage of output.

The interest rate series for Argentina are constructed by adding the quarterly EMBI+ spreads from JP Morgan's EMBI+ database on Argentinean foreign-currency-denominated bonds to the interest rate of five-year U.S. treasury bonds. Interest rates are reported as percentages in annual terms.

We calibrate the parameters β, κ_1, κ_2, and θ in our model to match key moments for the Argentinean economy. We set the borrower's discount factor β to target an annual frequency of default of 3 percent. The calibrated value for β is 0.6335. Yue (2010) uses a discount factor of 0.74, and Aguiar and Gopinath (2006) uses 0.80. High impatience on the borrower's side is required to generate default events with the targeted frequency.[35]

Low discount factors for emerging economies may capture other relevant noneconomic aspects, not modeled explicitly in our economy, such as political uncertainty, as analyzed by Amador (2003), Cuadra and Sapriza (2008), D'Erasmo (2011), and Chang (2010).

We select the output cost parameters κ_1 and κ_2 to match the average debt level of 46 percent of GDP for Argentina and the spreads volatility of 5.38 percent.[36]

Regarding the degree of model uncertainty in our economy, we take the following strategy: we first set the penalty parameter θ to match the average bond spreads of 10.25 percent

[34] Pitchford and Wright (2012) reports an average 6.5-year delay in debt restructuring after 1976.

[35] The value for the discount factor needed to match the default frequency is low as a consequence of the output cost specification rather than because of the presence of doubts about model misspecification. Chatterjee and Eyingungor (2012) requires a similar value to induce a sufficient number of defaults with one-period bonds. Their quantitative results suggest that introducing long-term debt enables them to attain similar asset-pricing implications keeping the same default frequency, but with a significantly higher discount factor. We find modeling a more realistic maturity structure an interesting extension left for future research

[36] The foreign government debt to output ratio of 46 percent for Argentina is taken from INDEC for the period from 1994:Q4 to 2001:Q4.

observed in the data for Argentina. As suggested in Section 6, the value of θ is itself not necessarily very informative of the amount of distortion in lenders' perceptions about the evolution of y. The impact of the value of θ on probability distortions is context-specific.[37] To better interpret our results, we provide another statistic, the detection error probabilities (DEP), commonly used in the robust control literature.[38] In our economy lenders are concerned about alternative models that they find hard to differentiate statistically from one another, given the available series of output data. To measure how close two competing models are and, therefore, how difficult it is to distinguish between them, we use DEP. DEP gauges the likelihood of selecting the incorrect model when discriminating between the approximating and the worst-case model using likelihood ratio tests; for details on its computation see Section 6. The lower the value of DEP, the more pronounced is the discrepancy between these two models. If they are basically identical, they are indistinguishable and hence the DEP is 0.50. In contrast, if the two models are perfectly distinguishable from each other, the DEP is 0. Barillas, Hansen, and Sargent (2009) suggests 20 percent as a reasonable threshold, in line with a 20-percent Type I error in statistics. In our model the DEP implied by our calibrated θ is only 35 percent, which implies that around one third of the time the detection test indicates the wrong model. This value is therefore quite conservative, suggesting that only a modest amount of model uncertainty is sufficient to explain the high average bond spreads observed in the data.

Computational algorithm. The model is solved numerically using value function iteration. To that end, we apply the discrete state space (DSS) technique. The endowment space is discretized into 101 points and the stochastic process is approximated to a Markov chain, using Tauchen and Hussey's (1993) quadrature-based method.[39]

To compute the business cycle statistics, we run $3,000$ Monte Carlo (MC) simulations of the model with $1,000$ periods each.[40] As in Arellano (2008), to replicate the period for Argentina between default events, from 1983:Q3 to 2001:Q4, we consider 2,000 subsamples of 74 periods with access to financial markets, followed by a default event. We then compute the mean statistics and the 90-percent confidence intervals, across MC simulations, for these

[37]See Barillas, Hansen, and Sargent (2009) for a simple example with a random walk model and a trend stationary model for log consumption.

[38]See Anderson, Hansen, and Sargent (2003), Maenhout (2004), Barillas, Hansen, and Sargent (2009), Bidder and Smith (2011), and Luo, Nie, and Young (2012), for example.

[39]For bond holdings, we use 300 gridpoints to solve the model and no interpolation

[40]To avoid dependence on initial conditions, we pick only the last 200 periods from each simulation. The unconditional default frequency is computed as the sample mean of the number of default events in the simulations.

		Parameter	Value
Borrower			
	Risk aversion	σ	2
	Time discount factor	β	0.6335
	Probability of reentry	π	0.0385
	Output cost parameter	κ_1	-0.444
	Output cost parameter	κ_2	0.552
	AR(1) coefficient for y_t	ρ	0.945
	Std. deviation of ε_t	σ_ε	0.025
Lender			
	Robustness parameter	θ	0.448
	Time discount factor	γ	0.9832
	Constant for y	α	1.00

Table 1: Parameter values.

subsamples.

Solving sovereign default models using the discrete state space (DSS) technique may encounter convergence problems for some particular environments, as pointed out by Chatterjee and Eyingungor (2012). Since default is a discrete choice, it can occur that—under DSS technique—the debt policy rule is not continuous in the current state variables and prices. In turn, the discontinuity in the debt policy function with respect to bond prices translates into discontinuity in the lenders' Euler equation, which may lead to convergence problems.[41]

To handle this technical complication we propose an approach based on the introduction of an i.i.d. preference shock, denoted as $\nu \sim F_\nu$, in line with McFadden (1981) and Rust (1994). We add ν only to the autarky utility of the borrower when it decides whether to default on the debt; but we do not add any shock when the economy is already in financial autarky.[42]

The shock ν could be interpreted as an error of an agent who intends to behave according to a certain payoff, but who incorrectly calculates the payoff by adding a noise.

We call the economy with the shock, the *perturbed economy*; we relegate the formal details of this new economy to Appendix F. Also in the appendix we show the following result:[43]

[41] For more details, see Chatterjee and Eyingungor (2012).

[42] Alternatively, we could have added the preference shock to the repayment utility of the borrower when making the default decision; due to the nature of the shock distribution, the results would remain unchanged.

[43] See Lemma F.3 in Appendix F for a formal statement.

Lemma 4.1. *As the distribution of ν converges to a mass point at zero, if the equilibrium in the perturbed economy converges, it does so to the equilibrium in the economy without the shock (that is, the economy described in Section 2.7).*

The result is an adaptation of Proposition 4 in Doraszelski and Escobar (2010) to our setting. It basically states that as the distribution of ν becomes more concentrated around zero, then the limit of the equilibrium in the perturbed economy (if it exists), is an equilibrium described in Section 2.7.

For our numerical results, we postulate F_ν to be a logistic(h) distribution, that is, $F_\nu(v) = \frac{1}{1+\exp\{-h^{-1}v\}}$, where the parameter h controls for the variance of the distribution. Note that as $h \to 0$, F_ν converges to a mass point at zero.[44] Under this particular choice, it follows that the Bellman equation of the borrower when it decides to repay is given by

$$V_R(y_t, B_t) = \max_{B_{t+1}}\{u(c_t) + \beta \int_{\mathbb{Y}} \left(h \log \left(1 + \exp \left\{ \frac{V_A(y_{t+1}) - V_R(y_{t+1}, B_{t+1})}{h} \right\} \right) + V_R(y_{t+1}, B_{t+1}) \right) f(y_{t+1}|y_t) dy_{t+1} \}.$$

The value function V_A changes in a similar fashion. The default policy function turns out to be a probability,

$$(y_t, B_t) \mapsto \frac{1}{1 + \exp\{h^{-1}(V_A(y_t) - V_R(y_t, B_t))\}}. \tag{10}$$

To our knowledge, in the sovereign default literature, there is only one other paper, by Chatterjee and Eyingungor (2012), that uses a similar approach to deal with a problem of the same nature. Their paper proposes an accurate method based on introducing a continuous i.i.d. shock to output. In contrast to Chatterjee and Eyingungor (2012), however, we are able to show the limiting properties of the equilibria of the economy with preference shock. Also, our choice of distribution for ν delivers closed forms for continuation values with access to financial markets and for the default indicator. This has important implications in terms of computation, since it does not rely on constructing an additional grid for the i.i.d. shock, which could slow down the algorithm considerably, and could introduce additional numerical/approximation errors.

Finally, it is worth pointing out that in our benchmark simulations we employed the solutions with preference shock as starting values for the iterations on the Bellman equations without it. Hence, no default event in our results is driven by the dynamics of the preference

[44] See Assumption F.1 in the Appendix F for the precise definition of convergence.

shock.

Output costs and implications. The choice of specification for the output costs of default is key for matching some business cycle moments.

In our calibration, we have $\kappa_1 < 0$, which implies that there are no output costs for realizations $y < \kappa_2/\kappa_1$, and the output costs as a fraction of output increase with y for $y > \kappa_2/\kappa_1$. In this sense, our output costs are similar to those in Arellano (2008), both of which have significant implications for the dynamics of debt and default events in the model. As explained before, when output is high, there is typically less default risk, bond returns are low, and there is more borrowing. For low levels of output, the costs of default are lower, hence, the default risk is higher, and so are the bond returns. If the borrower is hit by a sufficiently long sequence of bad output realizations, it eventually finds it optimal to declare default.

As noted by Chatterjee and Eyingungor (2012), our functional form for output costs has an important advantage over those of Arellano (2008) for the volatility of bond spreads. The difference in the spreads variability is even greater once we allow for model uncertainty. In Arellano (2008) output costs as a fraction of output vary significantly with output, and so do the default incentives. Hence, the default probability is very sensitive to the endowment realizations y.

Quantitatively, in that specific framework, spreads volatility can be considerably reduced when using alternative computational methods to solve the model numerically, as shown by Hatchondo, Martínez, and Sapriza (2010). The volatility of bond spreads rises significantly when we introduce doubts about model misspecification. As discussed in Section 3.1, probability distortions largely depend on the default risk under the approximating model and therefore are also very sensitive to endowment realizations y. Therefore, two different factors determine the significant variability of the distorted conditional probability, boosting the volatility of bonds spreads. In addition to the high sensitivity of the default probability under the approximating model, as in the standard rational expectations equilibrium of Arellano (2008), here we have the beliefs' distortions playing in the same direction, by slanting probabilities even more toward the range of endowment realizations in which default occurs. For this reason, we consider an alternative output cost structure from Arellano (2008), given by $\max\{0, \kappa_1 y + \kappa_2 y^2\}$. In this case, the output loss as a proportion of output is less responsive to fluctuations of y. It therefore yields a lower sensitivity of default probabilities to y, reducing at the same time the spreads volatility.

Statistic	Data	Baseline Model		Our Model	
Mean$(r - r^f)$	10.25	3.58	(1.7,5.8)	10.23	(6.7,15.2)
Std.dev.$(r - r^f)$	5.58	6.36	(2.4,10.2)	5.63	(3.3,9.3)
mean$(-b/y)$	46	5.95	(0.8,11.0)	44.84	(31.9,58.9)
Std.dev.(c)/std.dev. (y)	1.10	1.10		1.63	
Std.dev.(tb/y)	1.75	1.50	(0.7,1.8)	5.94	(4.9,7.0)
Corr(y, c)	0.98	0.97	(0.95,0.99)	0.76	(0.7,0.8)
Corr$(y, r - r^f)$	-0.88	-0.29	(-0.6,0.3)	-0.73	(-0.8,-0.6)
Corr$(y, tb/y)$	-0.64	-0.25	(-0.4,-0.07)	-0.23	(-0.4,-0.1)
Kurtosis$(r - r^f)$	4.14	22.40	(2.2,45)	4.97	(2.5,10.2)
Skewness$(r - r^f)$	1.02	3.98	(1.0,6.0)	1.20	(0.5,2.3)
Default frequency (annually)	3.00	3.00	(0.0,8.2)	3.00	(0.0,6.1)

Table 2: Business Cycle Statistics for our Model, the Data and the Baseline Model. The 90-percent confidence intervals generated by the MC simulations are reported in parentheses.

5 Quantitative Results

Table 2 reports the moments of our benchmark model and in the data. For comparison purposes, it also shows the corresponding moments for the economy of Arellano (2008); we call this model the "baseline model."[45]

Overall, the model matches several business cycle features of the Argentinean economy. First, we can replicate salient features of the bond spreads dynamics. We can account for all the average bond spreads observed in the data, as well as their volatility, matching at the same time the historical annual frequency of default of 3 percent. By introducing doubts about model misspecification, our model can explain the average bond spreads of 10.25 percent in the data, which is roughly three times as high as the 3.58 percent obtained by the baseline model. In fact 10.25 percent is outside its 90 percent confidence interval. In our environment, risk-neutral lenders charge an additional uncertainty premium on bond holdings to get compensated for bearing the default risk under the worst-case density for output. In turn, their *perceived* conditional probability of default next period—while having access to financial markets—is on average 2.2 percent per quarter, while the actual one is only 0.9 percent. Lenders' distorted beliefs about the evolution of the borrowing economy enable us to achieve the challenging goal of simultaneously matching the low sovereign default frequency and the high average level (and volatility) of excess returns on Argentinean bonds exhibited in the data. While the penalty parameter θ is calibrated to match this moment

[45]Even though Arellano (2008) targets different moments in the calibration, we think that comparing our business cycle statistics to hers may help to highlight the main contributions of our model.

in the data, it is surprising that introducing model uncertainty is a clear and sufficient mechanism to make this possible. Interestingly, to obtain this, we require only a quite limited amount of model uncertainty. Indeed, we need on average smaller deviations of lenders' beliefs to explain the spread dynamics than those used in the equity premium literature.[46] We consider this an important contribution of our paper.[47]

To our knowledge, only two other papers have been able to match the average bond spreads observed in the data: Chatterjee and Eyingungor (2012) and Hatchondo, Martínez, and Padilla (2010). In their models, the borrower issues long-term debt. In contrast with models with one-period bonds, their models are able to generate positive spreads even when there is no default risk in the near future. The authors, however, reproduce the average high spreads for Argentina at the cost of roughly doubling the default frequency to around 6 percent annually.[48]

Furthermore, we want to stress that we replicate this feature of the bond spreads in a general equilibrium framework. Arellano (2008) and some recent studies on long-term debt, such as Arellano and Ramanarayanan (2012) and Hatchondo, Martínez, and Padilla (2010), have been able to account for this feature but by assuming an ad hoc functional form for the stochastic discount factor, which depends on the output shock to the borrowing economy. Our paper can be seen as providing microfoundations for such a functional form.

Also, the introduction of plausible degrees of risk aversion on the lenders' side with time-separable preferences has been shown to be insufficient to recover the high spreads observed in the data. With constant relative risk aversion, as in Lizarazo (2010), matching high spreads calls for a very large risk aversion coefficient, in line with Mehra and Prescott (1985), and implausible risk-free rates, as pointed out by Weil (1989) in the context of studies on the equity premium. In Appendix C we show simulations that quantify these facts for this setting.[49]

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[46]To explain different asset-pricing puzzles, Maenhout (2004), Drechsler (2012), and Bidder and Smith (2011) require detection error probability in the range between 10 and 12 percent. Barillas, Hansen, and Sargent (2009) need even lower values to reach the Hansen and Jagannathan (1991) bounds.

[47]It is worth noting that while we assume no recovery on defaulted debt in the model—which in equilibrium pushes up the bond returns—there is room to increase the amount of model uncertainty (that is, decrease θ) within the plausible range, and thus we could still account for the bond spread average level if any mechanism of debt restructuring with subsequent haircuts were introduced.

[48]In addition, the period analyzed by Chatterjee and Eyingungor (2012) is different, spanning from 1993 to 2001, during which the bond spreads for Argentinean debt were on average only 8.15 percent, instead of the 10.25 percent in our case. Hatchondo, Martínez, and Padilla (2010) consider a period with slightly lower average spreads.

[49]Borri and Verdelhan (2010) has studied the setup with positive co-movement between lenders' consumption and output in the emerging economy in addition to time-varying risk aversion on the lenders' side. To generate endogenous time-varying risk aversion for lenders, they endow them with Campbell and Cochrane

Second, our model can deliver strongly countercyclical bond spreads. When output goes up, the default risk typically decreases, hence, lenders demand lower returns on their bond holdings. With uncertainty-averse lenders, the *perceived* default risk—the relevant one for pricing the asset—decreases even further, amplifying the effect. The correlation between the borrower's output and bond spreads is below that observed in the data (at least with 90 percent confidence), but still higher than in the baseline model. It should not be surprising anyhow that even in these cases interest rates are not as countercyclical as observed in the data since we consider an endowment economy and there is no feedback from interest rates into output.

Third, we also report the kurtosis and skewness in spreads observed in the data for Argentina. In the data, the spreads exhibit a kurtosis of 4.14 and positive skewness of 1.02; both indicate that the data have longer, fatter tails, especially for positive values.[50] For the sake of comparison, we computed these moments for the spreads for two developed open economies, Australia and Canada. The kurtosis is 2.46 and 2.67, respectively, and the skewness is 0.74 and 0.34, respectively. The difference is large and reflects the lack of high positive values of the spread before the eve of the default event, present in Argentina.[51]

Our model overshoots the kurtosis by roughly 20 percent, whereas the baseline model does so by almost 450 percent. Our model also overshoots the skewness by roughly 17 percent, whereas the baseline model does so by approximately 290 percent. Both models, however, include both data points in their corresponding 90-percent confidence intervals.

In order to shed more light on the behavior of the spreads, especially at the upper tail, we report in Table 3 different percentiles. In all cases, our model's 90-percent confidence intervals contain the value observed in the data, and the average across MC simulations is very close to the one observed in the data. The baseline model, however, yields percentiles, in terms of both the average across MC simulations and the confidence intervals, which are considerably below the values observed in the data. Finally, we note that the median is always lower than the average in the data and in the models, due to the presence of large peaks because of the default events. These results show how our model is able to match the average level, volatility, and countercyclicality of spreads, while not distorting other, higher

(1999) preferences with external habit formation. However, they find that even with these additional components average bond spreads generated by the model are far below those in the data. They report average bond spreads of 4.27 percent, for an annual default frequency of 3.11 percent.

[50]Ecuador exhibits a similar pattern to the Argentinean one, with kurtosis of 4.35 and skewness of 1.39.

[51]For Australia we use 90-day commercial bills for 1983:Q2-2001:Q4; for Canada we use 90-day prime corporate paper for 1983:Q2-2001:Q4; finally, for Ecuador we take the series from Arellano (2008), for 1983:Q1-1989:Q1. Data are available upon request.

Statistic	Data	Baseline Model		Our Model	
$Q_{0.10}(r - r^f)$	4.13	0.04	(0.0,0.4)	4.41	(2.5,6.8)
$Q_{0.25}(r - r^f)$	6.15	0.35	(0.0,0.6)	6.23	(4.0,9.6)
$Q_{0.50}(r - r^f)$	9.31	2.21	(0.0,4.8)	9.01	(5.8,13.2)
$Q_{0.75}(r - r^f)$	13.67	4.57	(0.7,5.5)	12.92	(8.3,19.6)
$Q_{0.90}(r - r^f)$	17.05	6.03	(4.8,7.2)	17.69	(11.3,26.9)

Table 3: Quantiles of Spreads for our Model, the Data and the Baseline Model. $Q_\alpha(r - r^f)$ denotes the α-th quantile. The 90-percent confidence intervals generated by the MC simulations are reported in parentheses.

moments.

Fourth, the model can generate considerable levels of borrowing, consistent with levels observed in the data. High output costs of default jointly with a low probability of regaining access to financial markets imply a severe punishment to the borrower in case it defaults.[52] Consequently, higher levels of indebtness can be sustained in our economy.

Fifth, the model reproduces *qualitatively* standard empirical regularities of emerging economies: higher volatility of consumption relative to output, a strong correlation between consumption and output, and volatility and countercyclicality of net exports. Quantitatively, however, our model performs worse than that of Arellano (2008) along these dimensions. The discrepancy is mostly a result of the difference in equilibrium debt levels between these two economies. Since in both economies the borrower can issue only one-period bonds, to maintain the same level of indebtness b, the borrower needs to refinance the bonds at the new price $q(y, b)$, which is very sensitive to fluctuations in output y. Large and volatile capital outflows for interest payments translate into excess volatility of consumption and net exports.

Different degrees of concern about model misspecification. In Table 4 we report some business cycle statistics from the simulations of our model for different degrees of model uncertainty and no risk aversion on the lenders' side. We start with no fears about model misspecification, that is, $\theta = +\infty$, and we lower the penalty parameter to 0.25, for which we obtain a detection error probability of 27.55 percent. As expected, when we reduce the value of θ, we observe that the frequency of default goes down. To keep it at the historical level of 3 percent per year, we make the borrower more impatient by adjusting β downwards.

In the comparison across models, which typically differ in several dimensions along their parametrization and assumed functional forms, it may be hard to identify which key

[52]For our parametrization, the loss in output following a default is weakly larger in our economy than in Arellano (2008) for any endowment realization y.

Statistic	$\theta = +\infty$	$\theta = 5$	$\theta = 1$	$\theta = 0.75$	$\theta = 0.5$	$\theta = 0.25$
Mean$(r - r^f)$	4.19	4.71	6.87	7.84	9.65	15.54
Std.dev.$(r - r^f)$	3.31	3.40	4.34	4.64	5.34	8.74
Mean$(-b/y)$	77.05	72.35	57.95	53.33	46.63	33.80
Std.dev.(c)/std.dev.(y)	1.82	1.80	1.74	1.70	1.63	1.46
Std.dev.(tb/y)	7.15	7.08	6.71	6.49	6.05	4.89
Corr(y, c)	0.74	0.73	0.74	0.74	0.76	0.80
Corr$(y, r - r^f)$	−0.60	−0.59	−0.67	−0.68	−0.71	−0.74
Corr$(y, tb/y)$	−0.28	−0.27	−0.25	−0.24	−0.23	−0.22
DEP	0.5000	0.4935	0.4303	0.3990	0.3660	0.2755
Default frequency (annually)	3.00	3.00	3.00	3.00	3.00	3.00

Table 4: Business Cycle Statistics for Different Degrees of Robustness

ingredient is driving each difference in the simulated statistics. Table 4 helps us highlight the contribution of model uncertainty by showing how the dynamics of relevant macro variables vary in the same environment as we increase the preference for robustness.

The first finding is that both the mean and the standard deviation of bond spreads increase with the lenders' concerns about model misspecification. For the same default frequency, as θ decreases, two opposing forces interact. On one hand, for the same allocations, a greater degree of concern about model misspecification tends to push higher the probability distortions associated with low utility states for the lender. On the other hand, as bond prices are pulled down, the borrower responds by borrowing less. A smaller quantity of bonds issued by the borrower leads to smaller potential losses for the lenders in the event of a default, implying smaller spreads over the lenders' continuation values across states, which in turn lower the probability distortions. From Table 4, it is clear that the first effect prevails, and bond spreads climb as θ goes down.

As the gap in probability distortions between the default and nondefault states widens, the distorted probability becomes more volatile and sensitive to fluctuations in output. As a result, the volatility and countercyclicality of bond spreads increases.

Second, as mentioned before, the level of indebtness falls significantly. As the demand for bonds decreases, price and quantity demanded go down. Average debt-to-output ratios for the borrower move from around 77 percent to slightly below 34 percent.

Third, capital outflows and consumption become less volatile, due to smaller variability in the interest repayments. At the same time, output and consumption become more correlated with each other. Even though bond prices become more volatile, the reduction in the mean and variance of the debt level leads to less volatile interest repayments.

5.1 A Graph for the Argentinean Case

In order to showcase the dynamics generated by our model, we perform the following exercise. We input into the model the output path observed in Argentina for 1994:Q2 to 2001:Q4. Given this and an initial level of debt, our model generates a time series for the annualized spread and for one-step-ahead conditional probabilities of default under both the approximating and distorted models. The top panel in Figure 3 shows the output path, jointly with the time series for bond spreads exhibited in the data and delivered by our model. For comparison, we also plot the spreads generated by the baseline model. The bottom panel displays the conditional default probabilities according to our model.

We can see that our model does a better job matching the actual spreads than the baseline model does. It is interesting to note that our model predicts that default occurs in 2001:Q4; the same quarter in which Argentina declared its default. The difference between the spreads generated by the two models can be largely explained by the behavior of one-step-ahead conditional probabilities of default. Our model generates, on average, a higher (subjective) probability of default for the next period. Perhaps more importantly, the wedge between the two probabilities is greater when output is low (and default is more likely next period), for example, see results for 1994:Q4 to 1995:Q4 and 2001:Q2. Moreover, note that during 1995, the distorted default probability of a default next period fluctuates between 2.29 and 2.65 percent and in 2001:Q3 it hits 3.09 percent. These probabilities are over 3 times larger than the average quarterly default frequency of 0.74 percent and more than twice as large as the one-step-ahead conditional probability under the approximating model.

6 Detection Error Probabilities

In this section we use detection error probabilities (DEP) to measure the amount of model uncertainty in this economy and interpret the value of penalty parameter θ in the calibration. For this we follow Anderson, Hansen, and Sargent (2003), Maenhout (2004), and Barillas, Hansen, and Sargent (2009) among others. We also propose an alternative way of interpreting θ.

The impact of the parameter θ on the probability distortions and other endogenous quantities of the model is environment specific. It depends on other parameters of the model and on the parametrization itself. However, from the Bellman equations in Section 2.6, it is

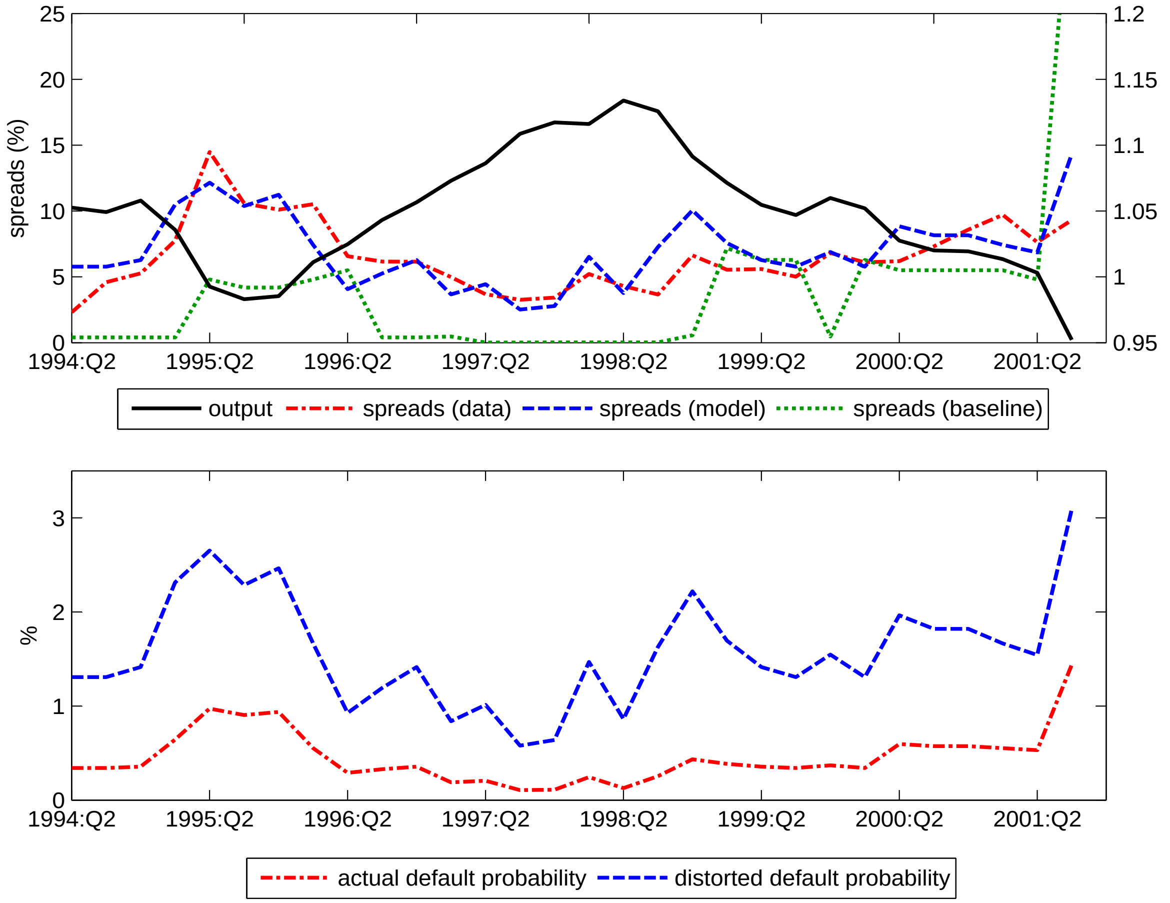

Figure 3: Times series for Argentina and models. Top panel: Output for Argentina; spreads generated by our model and the baseline model; and actual spreads (measured by the EMBI+). Bottom panel: One-step-ahead conditional probabilities of default under the distorted model and the approximating model.

easy to see that the higher θ is, the lower the concerns for model misspecification are, and in turn, the closer the distorted and approximating models are. As explained in Section 4, we use DEP to formalize this relationship, in line with Barillas, Hansen, and Sargent (2009).

Let $L_{A,T}$ and $L_{\theta,T}$ be the likelihood functions corresponding to the approximating and distorted models for $(Y_t)_{t=1}^{T}$, respectively. Let $\Pr(\cdot \mid L_A)$ and $\Pr(\cdot \mid L_\theta)$ be the respective probabilities over the data, generated under the approximating and distorted models. Let $p_{A,T}(\theta) \equiv \Pr\left(\log\left(\frac{L_{\theta,T}}{L_{A,T}}\right) > 0 \mid L_A\right)$ and $p_{D,T}(\theta) \equiv \Pr\left(\log\left(\frac{L_{\theta,T}}{L_{A,T}}\right) < 0 \mid L_\theta\right)$, let the DEP be obtained by averaging $p_{D,T}(\theta)$ and $p_{A,T}(\theta)$:

$$DEP_T(\theta) = \frac{1}{2}\left(p_{A,T}(\theta) + p_{D,T}(\theta)\right).$$

If the two models are very similar to each other, mistakes are likely, yielding high values of $p_A(\theta)$ and $p_D(\theta)$; the opposite is true if the models are not similar.[53]

The aforementioned quantities can be approximated by means of simulation. We start by setting an initial debt level and endowment vector. We then simulate time series for output for $T' = 1,000 + T$ periods (quarters), where $T = 74$.[54] The process is repeated 1,000 times.

For each time-series realization, we construct $L_{A,T}$ and $L_{\theta,T}$. We then compute the percentage of times the likelihood ratio test indicates that the worst-case model generated the data (when the data were generated by the approximating model); we denote this as $p_{A,T}(\theta)$.[55] Similarly, we use $p_{D,T}(\theta)$ to denote the percentage of times the likelihood ratio test indicates that the approximating model generated the data (when the data were generated by the distorted model). Finally, we compute $DEP_T(\theta) \equiv \frac{1}{2}\left(p_{A,T}(\theta) + p_{D,T}(\theta)\right)$.

For a *given* number of observations (in our case 74), as $\theta \to +\infty$, the approximating and distorted models become harder to distinguish from each other and the detection error probability converges to 0.5. If instead they are distant from each other, the detection error probability is below 0.5, getting closer to 0 as the discrepancy between the models gets larger.

Following Barillas, Hansen, and Sargent (2009), we consider a threshold for the DEP of 0.2; values of $DEP_T(\theta)$ that are larger or equal are deemed acceptable. In our calibration, our DEP is above this threshold, since $DEP_T(\theta) = 0.356$. For this value of θ (and other parameters), $p_{A,T} = 0.345$ and $p_{D,T} = 0.367$. So the weight of 0.5 does not play an important

[53]The weight of one-half is arbitrary; see Barillas, Hansen, and Sargent (2009) among others. Moreover, as the number of observations increases, the weight becomes less relevant, since the quantities $p_{A,T}$ and $p_{D,T}$ get closer to each other; as shown in Figure 4

[54]$T = 74$ is chosen to replicate the number of periods used in the calibration. For both models, we ignore the first 1,000 observations in order to avoid any dependence on our initial levels of debt and endowments.

[55]In the case of $L_A = L_\theta$, we count this as a false rejection with probability 0.5.

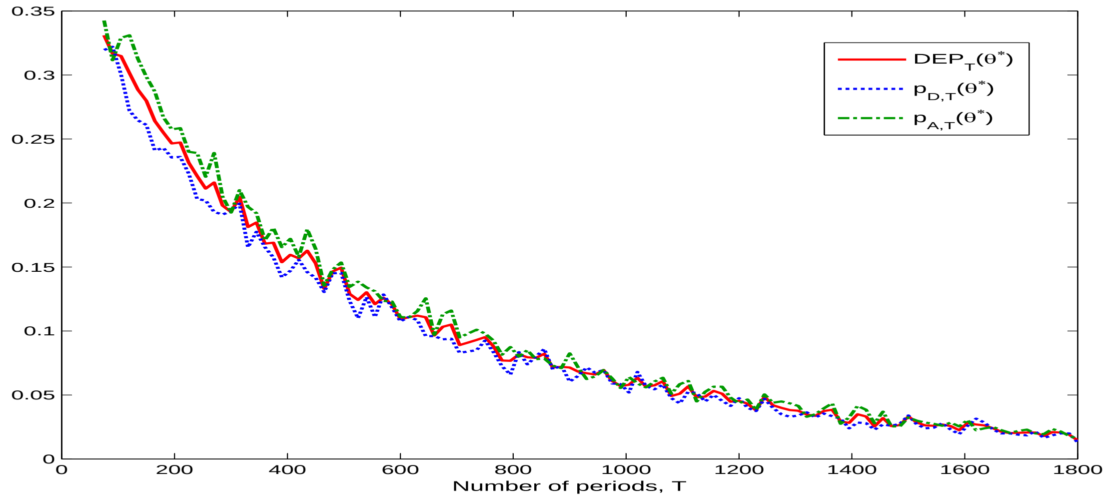

Figure 4: Detection error probability and its components as function of the number of periods, T, for our calibrated economy.

role.

We conclude the section by proposing an alternative view for interpreting θ. This is based on the following observation: for any *fix* finite θ (for which $L_{\theta,T}$ exists), $L_{A,T} \neq L_{\theta,T}$ with positive probability; thus, as the number of observations increases, $p_{T,k}(\theta) \to 0$ for $k = \{A, D\}$. Therefore, for a given level of θ and an a priori chosen level $\alpha \in (0, 1)$, which does not depend on θ, we can define $T_{\alpha,\theta} \equiv \max\{T : p_T(\theta) = \alpha\}$, as the maximum number of observations before $DEP_T(\theta)$ falls bellow α. A heuristic interpretation of this number is that the agents need at least $T_{\alpha,\theta}$ observations to be able to distinguish between the two models at a certainty level of α. The higher this number, the harder it is to distinguish between the models.

Figure 4 plots $\{p_{T,A}(\theta), p_{T,D}(\theta), DEP_T(\theta)\}_{T=75}^{1,800}$ for the given value of θ in our calibration. For a level of $\alpha = 0.2$, we see that $T_{\alpha,\theta} \approx 300$. That is, one needs approximately 4.5 times our sample of 74, in order to obtain a level of $\alpha = 0.2$ for $DEP_T(\theta)$ and consequently claim that these models are sufficiently different from each other, according to this criterion.

Under both interpretations—the one using the threshold of 0.2 in Barillas, Hansen, and Sargent (2009) or looking at $T_{\alpha,\theta}$—the probability distortions associated with this value of θ are typically small. This follows from the fact that the discrepancy between the approximating and distorted models is largest for values of the state where default occurs; see Figure 1. Since default is a "rare event" (for example, in our calibration, it occurs approximately three times every 100 years), most of the time the discrepancy between the the approximating and distorted models is small.

7 Conclusion

This paper accounts for the high bond spreads observed for Argentina, while keeping the default frequency at historical levels and at the same time matching the standard empirical regularities of these economies. We achieve this by introducing fears about model misspecification on the lenders' side. Lenders in this economy fear that the probability model governing the evolution of the endowment of the borrower is misspecified. They contemplate a set of alternative probability models and seek decision rules that perform well across them. To compensate for a risk and uncertainty-adjusted probability of default, they demand higher returns on their bond holdings.

We also propose an approach to tackle convergence issues when solving the model numerically using the DSS technique. This approach is based on the introduction of an i.i.d. preference shock for the borrower, drawn from a logistic distribution.

In future research we plan to extend the framework to account for a process for the borrower's endowment with stochastic trend. As pointed out by Aguiar and Gopinath (2007), shocks to trend growth—rather than transitory fluctuations—are the primary source of fluctuations in emerging markets. As in the long-run risk literature, we would like to explore the implications that substantial volatility in trend growth and the inability to clearly identify transitory from trend shocks have on the dynamics of prices and allocations in emerging economies.

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A Moments of approximating and distorted densities.

Table 5 presents the computed moments for the approximating and distorted conditional densities of next-period y_{t+1}, given current y_t and bond holdings B_t. As shown in Figure 1, the current endowment level y_t is set to two conditional standard deviations below its unconditional mean, and the bond holdings B_t is given by the 25th percentile of its unconditional distribution in the simulations.

Moment	Approximating Model	Distorted Model
Mean(y_{t+1})	0.9551	0.9531
Std.dev.(y_{t+1})	0.0239	0.0254
Skewness(y_{t+1})	0.0752	−0.0594
Kurtosis(y_{t+1})	3.0100	2.9846

Table 5: Moments for the Approximating and Distorted Conditional Densities.

By the Law of Large Numbers, the moments for the approximating model are essentially the same as the corresponding "population" moments of the lognormal distribution. Regarding the distorted model, all moments except skewness are very close to those of the approximating model. This follows from the fact that the distorted model puts more probability mass on low realizations of output, y_{t+1}, where default is optimal for the borrower, as illustrated in Figure 1. It does so by taking probability mass away from the repayment interval, but this change is small and it renders the location and dispersion of the density relatively unaffected.

B Correlation between $m^*_{R,t+1}$ and δ_{t+1}.

In Figure 5, we plot the correlation between $m^*_{R,t+1}$ and δ_{t+1}, first as a function of current y_t given debt holdings B_t, and then as a function of debt B_t, fixing current endowment y_t.[56,57] The shaded area corresponds to the interval where default is optimal in the current state. The payoff δ_{t+1} is not defined over that interval, as financial markets may be closed to the borrower, in which case there would be no trading of the risky bond. In the nonshaded area, for which there is borrowing in the current period, the correlation is always negative.

C Robustness Check: Risk aversion with time-additive, standard expected utility.

As displayed in Table 6, plausible degrees of risk aversion on the lenders' side with standard time-separable expected utility, are not enough to generate sufficiently high bond spreads while keeping the default frequency as observed in the data.

[56] We plot correlation, not covariances because the former provides a notion of magnitude since it is in $[-1, 1]$.

[57] For the left-side panel, B_t is set to the 25th percentile of its unconditional distribution in the simulations. For the right-side panel, y_t is given by its stationary level.

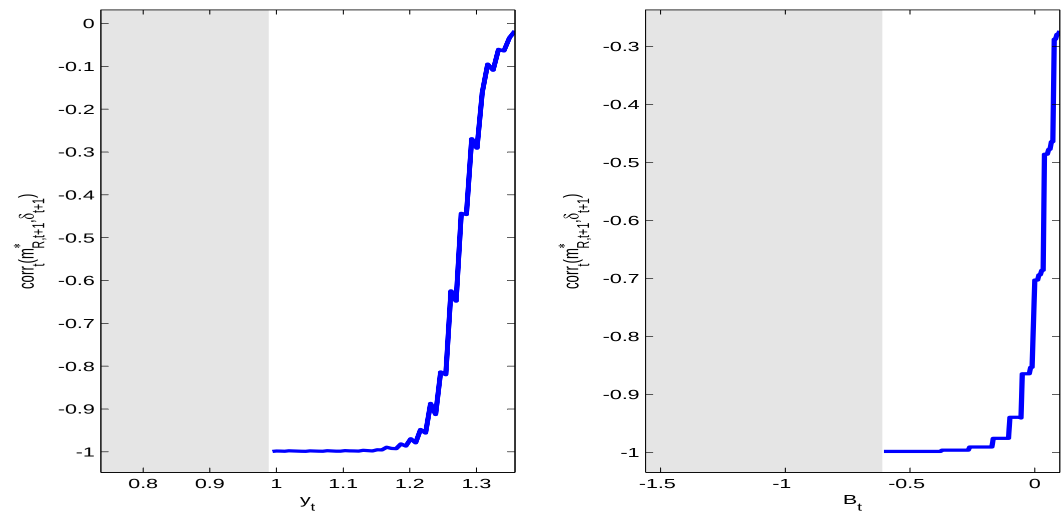

Figure 5: Conditional correlation between δ_{t+1} and $m^*_{R,t+1}$.

We considered an exogenous stochastic process for the lenders' consumption given by

$$\ln c^L_{t+1} = \rho^L \ln c^L_t + \sigma^L_\varepsilon \varepsilon^L_{t+1},$$

where $\varepsilon^L_{t+1} \sim i.i.d.\mathcal{N}(0,1)$. Shocks ε^B_{t+1} and ε^L_{t+1} are assumed to be independent. We estimate the log-normal AR(1) process for c^L_t, using U.S. consumption data.[58][59]

Table 6 displays the business cycle statistics for different values of the lenders' coefficient of relative risk aversion, σ^L, ranging from 1 to 50, and no fears about model misspecification, that is, $\theta = +\infty$.[60] First, we observe that bond spreads increase on average and become more volatile with the value of σ^L. We find, however, that plausible degrees of risk aversion are not enough to generate sufficiently high bond spreads while keeping the default frequency as observed in the data. Setting σ^L equal to 5 generates average bond spreads of just 4.20 percent, less than half the value observed in the data. Notice that even a value of 50 is not sufficient to generate high bond spreads. This high value for the coefficient of risk aversion is sufficient to explain the equity premium puzzle in Mehra and Prescott (1985). In contrast with the economy considered there, the stochastic discount factor in our model would not typically vary inversely with the bond payoff, limiting the ability of the model to generate sufficiently high bond spreads. Second, given a stationary process for consumption in our model, the net risk-free rate decreases and turns negative for sufficiently high values of σ^L, while its volatility grows. Facing lower risk-free rates, the borrower reacts by borrowing

[58]Time series for seasonally adjusted real consumption of nondurables and services at a quarterly frequency are taken from the Bureau of Economic Analysis, in logs, and filtered with a linear trend. The estimates for parameters ρ^L and σ^L_ε are 0.967 and 0.025, respectively.

[59]See Shapiro and Pham (2006) for a detailed report about the characteristics of local and international holders of Argentinean debt during the 2001 default.

[60]For each value of σ^L, the discount factor for the borrower, β, is calibrated to replicate a default frequency of 3 percent annually.

Statistic	$\sigma^L = 1$	$\sigma^L = 2$	$\sigma^L = 5$	$\sigma^L = 10$	$\sigma^L = 20$	$\sigma^L = 50$
Mean$(r - r^f)$	4.18	4.20	4.20	4.25	4.37	5.36
Std.dev.$(r - r^f)$	3.66	3.72	3.63	3.69	3.90	4.48
Mean(r^f)	6.93	6.85	6.85	4.86	-0.52	-30.78
Std.dev.(r^f)	0.34	0.68	1.69	3.34	6.41	11.72
Mean$(-b/y)$	79.33	79.04	78.08	79.20	86.31	118.88
Std.dev.(c)/std.dev.(y)	1.84	1.83	1.84	1.87	2.05	3.78
Std.dev.(tb/y)	7.38	7.32	7.29	7.52	8.56	18.38
Corr(y, c)	0.74	0.73	0.73	0.73	0.71	0.57
Corr$(y, r - r^f)$	-0.55	-0.55	-0.56	-0.55	-0.54	-0.46
Corr$(y, tb/y)$	-0.28	-0.28	-0.28	-0.28	-0.29	-0.30
Default frequency (annually)	3.00	3.00	3.00	3.00	3.00	3.00

Table 6: Business Cycle Statistics for Different Degrees of Risk Aversion.

more. The debt-to-output ratio increases by 50 percent as σ^L goes from 1 to 50. Finally, interest repayments become very sensitive to fluctuations in the lenders' consumption. As a result, both consumption and net exports become more volatile.

D Recursive Formulation of the Problem of the "Minimizing Agent"

In this section, we show that the principle of optimality holds for the problem of the "minimizing agent."

Let $\mathbf{c}^L$ be a consumption plan, that is, an stochastic process such that $c_t^L(y^t)$ is measurable with respect to $\mathcal{Y}^t$. A feasible consumption plan is one that satisfies the budget constraint for each t.

We note that, since $\mathbb{B}$ is bounded, and, in equilibrium, $q_t \in [0, \gamma]$, any feasible consumption plan is bounded, that is, $|c_t^L(y^t)| \leq C < \infty$ a.s-$\mathcal{Y}^t$.

Definition D.1. *Given a feasible consumption plan $\mathbf{c}^L$, for each (t, y^t), we say functions $(t, y^t, \mathbf{c}^L) \mapsto \mathcal{U}_t(\mathbf{c}^L; y^t)$, satisfy the sequential problem of the "minimizing agent" (SP-MA) iff*[61]

$$\mathcal{U}_t(\mathbf{c}^L; y^t) = \min_{(m_{t+j+1})_j} \sum_{j=0}^{\infty} \gamma^j \int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \left(\frac{M_{t+j}}{M_t} \right) \{c_{t+j}^L + \theta\gamma\mathcal{E}(m_{t+j+1})\} P(dy^{t+j}|y_t),$$

where $M_t \equiv \prod_{\tau=1}^t m_\tau$.

[61]Note that, since $c_t^L \geq -C$ and $\theta\mathcal{E} \geq 0$, the RHS of the equation is always well defined in $[-Const, \infty]$, where $Const$ is some finite constant.

Definition D.2. *Given a feasible consumption plan $\mathbf{c}^L$, for each (t, y^t), we say functions $(t, y^t, \mathbf{c}^L) \mapsto \mathcal{U}_t(\mathbf{c}^L; y^t)$, satisfy the functional problem of the "minimizing agent" (FP-MA) iff*

$$\mathcal{U}_t(\mathbf{c}^L; y^t) = \min_{m_{t+1} \in \mathcal{M}} \left\{ c_t^L + \theta\gamma\mathcal{E}(m_{t+1}) + \gamma \int_{\mathbb{Y}} m_{t+1} \left(\mathcal{U}_{t+1}(\mathbf{c}^L; y^t, y_{t+1}) \right) f(y_{t+1}|y_t) dy_{t+1} \right\}. \tag{11}$$

Henceforth, we assume that in both definitions, the "min" is in fact achieved. If not, the definition and proofs can be modified by using "inf" at a cost of making them more cumbersome. We also assume that $\sum_{j=0}^{\infty} \gamma^j \int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \left(\frac{M_{t+j}}{M_t}\right) \{c_{t+j}^L + \theta\gamma\mathcal{E}(m_{t+j+1})\} P(dy^{t+j}|\cdot)$ is measurable with respect to $\mathcal{Y}^\infty$.

Lemma D.1. *For any feasible consumption plan $\mathbf{c}^L$,*

(a) If $(\mathcal{U}_t(\mathbf{c}^L; y^t))_{t,y^t}$ satisfies the SP-MA, then it satisfies the FP-MA.

(b) Suppose for each (t, y^t) there exist functions $(t, y^t, \mathbf{c}^L) \mapsto \bar{\mathcal{U}}_t(\mathbf{c}^L; y^t)$ that satisfy the FP-MA and

$$\lim_{T\to\infty} \gamma^{T+1} \int_{\mathbb{Y}^{T+1}} M_{T+1} \left(\bar{\mathcal{U}}_{T+1}(\mathbf{c}^L; y^T, y_{T+1}) \right) f(y_{T+1}|y_T) P(dy^{T+1}) = 0, \tag{12}$$

for all M_{T+1} such that $M_{T+1} = m_{T+1}M_T$, $M_0 = 1$ and $m_{t+1} \in \mathcal{M}$. Then $(\bar{\mathcal{U}}_t(\mathbf{c}^L; y^t))_{t,y^t}$ satisfy the SP-MA.

The importance of this lemma is that it suffices to study the functional equation (11).

Proof of Lemma D.1. Step 1. Before showing the desired results, we show that it suffices to perform the minimization over $(m_t)_t \in \mathbb{M}$, defined as

$$\mathbb{M} \equiv \left\{ (m_t)_t \colon m_t \in \mathcal{M} \cap \sum_{j=0}^{\infty} \gamma^j \int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \left(\frac{M_{t+j}}{M_t}\right) \mathcal{E}(m_{t+j+1})\} P(dy^{t+j}|y_t) \leq C_{C,\gamma,\theta}, \forall y^t \right\},$$

where $C_{C,\gamma,\theta} = 2\frac{C}{(1-\gamma)\theta\gamma}$. We do this by contradiction. Suppose that (m_t) solves the minimization problem in SP-MA, and , $\sum_{j=0}^{\infty} \gamma^j \int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \left(\frac{M_{t+j}}{M_t}\right) \mathcal{E}(m_{t+j+1}) P(dy^{t+j}|y_t) > C_{C,\gamma,\theta}$. Since consumption is bounded,

$$\begin{aligned}\mathcal{U}_t(\mathbf{c}^L; y^t) =& \sum_{j=0}^{\infty} \gamma^j \int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \left(\frac{M_{t+j}}{M_t}\right) \{c_{t+j}^L + \theta\gamma\mathcal{E}(m_{t+j+1})\} P(dy^{t+j}|y_t) \\ \geq& \sum_{j=0}^{\infty} \gamma^j (-C) \left\{ \int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \left(\frac{M_{t+j}}{M_t}\right) P(dy^{t+j}|y_t) + \theta\gamma \int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \left(\frac{M_{t+j}}{M_t}\right) \mathcal{E}(m_{t+j+1}) P(dy^{t+j}|y_t) \right\}.\end{aligned}$$

Note that

$$\int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \frac{M_{t+j}}{M_t} P(dy^{t+j}|y_t) = \int_{\mathbb{Y}^{t+j-1}|\mathbb{Y}^t} \frac{M_{t+j-1}}{M_t} \left\{ \int_{\mathbb{Y}} m_{t+j} P(dy_{t+j}|y_{t+j-1}) \right\} P(dy^{t+j-1}|y_t)$$
$$= \int_{\mathbb{Y}^{t+j-1}|\mathbb{Y}^t} \frac{M_{t+j-1}}{M_t} P(dy^{t+j-1}|\cdot) = ... = 1.$$

Hence,

$$\sum_{j=0}^{\infty} \gamma^j \int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \left(\frac{M_{t+j}}{M_t}\right) \{c^L_{t+j} + \theta\gamma\mathcal{E}(m_{t+j+1})\} P(dy^{t+j}|y_t)$$
$$\geq -\frac{C}{1-\gamma} + \theta\gamma \sum_{j=0}^{\infty} \gamma^j \int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \left(\frac{M_{t+j}}{M_t}\right) \mathcal{E}(m_{t+j+1}) P(dy^{t+j}|y_t).$$

By assumption, the second term is larger than $\theta\gamma C_{C,\gamma,\theta}$. Hence, the value for the minimizing agent of playing $(m_t)_t$ is bounded below by $-\frac{C}{1-\gamma} + \theta\gamma C_{C,\gamma,\theta}$. By our choice of $C_{C,\gamma,\theta}$,

$$-\frac{C}{1-\gamma} + \theta\gamma C_{C,\gamma,\theta} = \frac{C}{1-\gamma}.$$

Since $\mathcal{E}(1) = 0$, the RHS is larger than $\sum_{j=0}^{\infty} \gamma^j \int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \left(\frac{M_{t+j}}{M_t}\right) \{c^L_{t+j} + \theta\gamma\mathcal{E}(m_{t+j+1})\} P(dy^{t+j}|y_t)$ with $m_t = 1$ for all t. Therefore, we conclude that

$$\mathcal{U}_t(\mathbf{c}^L; y^t) > \sum_{j=0}^{\infty} \gamma^j \int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \left(\frac{M_{t+j}}{M_t}\right) \{c^L_{t+j} + \theta\gamma\mathcal{E}(m_{t+j+1})\} P(dy^{t+j}|y_t), \ for\ m_t = 1 \forall t.$$

But since $m_t = 1$ for all t is a feasible choice, this contradicts the definition of $\mathcal{U}_t(\mathbf{c}^L; y^t)$.

Step 2.
(a) From the definition of SP-MA and equation (2), it follows that

$$\begin{aligned}
\mathcal{U}_0(\mathbf{c}^L; y_0) = & \min_{(m_{t+1})_t} \Bigg\{ \{c^L_0 + \theta\gamma\mathcal{E}(m_1)\} \\
& + \gamma \sum_{t=1}^{\infty} \gamma^{t-1} \int_{\mathbb{Y}} M_1 \left(\int_{\mathbb{Y}^t|\mathbb{Y}} \frac{M_t}{M_1} \{c^L_t + \theta\gamma\mathcal{E}(m_{t+1})\} P(dy^t|y_1) \right) f(y_1|y_0) dy_1 \Bigg\} \\
= & \min_{(m_{t+1})_t} \Bigg\{ \{c^L_0 + \theta\gamma\mathcal{E}(m_1)\} \\
(\star)\ & + \gamma \int_{\mathbb{Y}} m_1 \left(\sum_{s=0}^{\infty} \gamma^s \int_{\mathbb{Y}^{s+1}|\mathbb{Y}} \frac{M_{s+1}}{M_1} \{c^L_{s+1} + \theta\gamma\mathcal{E}(m_{s+2})\} P(dy^{s+1}|y_1) \right) f(y_1|y_0) dy_1 \Bigg\} \\
\geq & \min_{m_1} \left\{ \{c_0 + \theta\gamma\mathcal{E}(m_1)\} + \gamma \int_{\mathbb{Y}} m_1 \left(\mathcal{U}_1(\mathbf{c}^L; y^1) \right) f(y_1|y_0) dy_1 \right\},
\end{aligned}$$

where the first inequality follows from the definition of $\mathcal{U}$. The step $(\star)$ follows from interchanging the summation and integral. To show that this is valid, let $H_n \equiv \sum_{s=0}^{n} m_1 \gamma^s \int_{\mathbb{Y}^s} \frac{M_{s+1}}{M_1} \{c^L_{s+1} +$

$\theta\gamma\mathcal{E}(m_{s+2})\}P(dy^{s+1}|\cdot)$. We note that

$$|H_n| \leq \sum_{s=0}^{\infty} m_1\gamma^s \left(\int_{\mathbb{Y}^{s+1}|\mathbb{Y}} \frac{M_{s+1}}{M_1} |c^L_{s+1}| P(dy^{s+1}|\cdot) + \int_{\mathbb{Y}^{s+1}|\mathbb{Y}} \frac{M_{s+1}}{M_1} \theta\gamma\mathcal{E}(m_{s+2}) P(dy^{s+1}|\cdot) \right)$$
$$\leq \sum_{s=0}^{\infty} m_1\gamma^s \left(C \int_{\mathbb{Y}^{s+1}|\mathbb{Y}} \frac{M_{s+1}}{M_1} P(dy^{s+1}|\cdot) + \int_{\mathbb{Y}^{s+1}|\mathbb{Y}} \frac{M_{s+1}}{M_1} \theta\gamma\mathcal{E}(m_{s+2}) P(dy^{s+1}|\cdot) \right),$$

where the second line follows because c^L_t is bounded. Since, $\int_{\mathbb{Y}^{t+j}|\mathbb{Y}^t} \frac{M_{t+j}}{M_t} P(dy^{t+j}|y_t) = 1$ for all t and j (see step 1), and $\sum_{s=0}^{\infty} \gamma^s \int_{\mathbb{Y}^{s+1}|\mathbb{Y}} \frac{M_{s+1}}{M_1} \theta\gamma\mathcal{E}(m_{s+2}) P(dy^{s+1}|\cdot) \leq C_{M,\gamma,\theta}$ by step 1 in the proof. Hence

$$|H_n| \leq m_1 \times Const.$$

for some $\infty > Const. > 0$ (it depends on (γ, θ, M)). Since the RHS is integrable, by the Dominated Convergence Theorem, interchanging summation and integration is valid.

The final expression actually holds for any state (t, y^t),

$$\mathcal{U}_t(\mathbf{c}^L; y^t) \geq \min_{m_{t+1}} \left\{ c^L_t + \theta\gamma\mathcal{E}(m_{t+1}) + \gamma \int_{\mathbb{Y}} m_{t+1} \left(\mathcal{U}_{t+1}(\mathbf{c}^L; y^t, y_{t+1}) \right) f(y_{t+1}|y_t) dy_{t+1} \right\}. \quad (13)$$

On the other hand, by definition of U,

$$\mathcal{U}_0(\mathbf{c}^L; y_0) \leq M_0\{c^L_0 + \theta\gamma\mathcal{E}(m_1)\} + \gamma \sum_{t=1}^{\infty} \gamma^{t-1} \int_{\mathbb{Y}} M_1 \left(\int_{\mathbb{Y}^{t-1}} \frac{M_t}{M_1} \{c^L_t + \theta\gamma\mathcal{E}(m_{t+1})\} P(dy^t|y_1) \right) f(y_1|y_0) dy_1$$

for any $(M_t)_t$ that satisfies the restrictions imposed in the text. In particular, it holds for $(M_t)_t$ where m_1 is left arbitrary and $(m_t)_{t\geq 2}$ is chosen as the optimal one. By following analogous steps to those before, it follows that

$$\mathcal{U}_0(\mathbf{c}^L; y_0) \leq \{c^L_0 + \theta\gamma\mathcal{E}(m_1)\} + \gamma \int_{\mathbb{Y}} m_1 \left(\mathcal{U}_1(\mathbf{c}^L; y^1) \right) f(y_1|y_0) dy_1,$$

for any m_1 that satisfies the restrictions imposed in the text; it thus holds, in particular, for the value that attains the minimum. Note that this holds for any (t, y^t), not just $(t = 0, y_0)$, that is,

$$\mathcal{U}_t(\mathbf{c}^L; y^t) \leq \min_{m_{t+1}} \left\{ c^L_t + \theta\gamma\mathcal{E}(m_{t+1}) + \gamma \int_{\mathbb{Y}} m_{t+1} \left(\mathcal{U}_{t+1}(\mathbf{c}^L; y^t, y_{t+1}) \right) f(y_{t+1}|y_t) dy_{t+1} \right\}. \quad (14)$$

Therefore, putting together equations (13) and (14), it follows that $(\mathcal{U}_t)_t$ satisfies the FP-MA.

(b) Let $(\bar{\mathcal{U}}_t)_t$ satisfy the FP-MA and equation (12). Then, by simple iteration it is easy to see that

$$\bar{\mathcal{U}}_0(\mathbf{c}^L; y_0) \leq \lim_{T\to\infty} \sum_{j=0}^{T} \gamma^j \int_{\mathbb{Y}^j} (M_j)\left\{c_j^L + \theta\gamma\mathcal{E}(m_{j+1})\right\} P(dy^j)$$
$$+ \lim_{T\to\infty} \gamma^{T+1} \int_{\mathbb{Y}^{T+1}} M_{T+1}\left(\bar{\mathcal{U}}_{T+1}(\mathbf{c}^L; y^T, y_{T+1})\right) P(dy^{T+1}).$$

The last term on the RHS is zero by equation (12), so $\bar{\mathcal{U}}_0(\mathbf{c}^L; y_0) \leq \mathcal{U}_0(\mathbf{c}^L; y_0)$ (where $\mathcal{U}$ satisfies the SP-MA). The reversed inequality follows from similar arguments and the fact that $\mathcal{U}_0(\mathbf{c}; y_0)$ is the minimum possible value.

The proof for (t, y^t) is analogous. Therefore, we conclude that any sequence of functions $(\bar{\mathcal{U}}_t)_t$ that satisfies FP-MA and (12), also satisfies SP-MA. □

E Proofs of Lemmas 3.1 and 3.2

Throughout this section, let $\{c^*, c^{L,*}, B^*, b^*, m_R^*, m_A^*, \delta^*, V_R^*, V_A^*, W_R^*, W_A^*\}$, Γ, and q^* be the recursive equilibrium defined in 2.4, where the endowment of the lender is *constant and given by* y^L.

The next lemma is a formal reformulation of Lemma 3.1. For this, we introduce some notation. Let $\mathbb{E}(\tilde{y}^L)$ be the set of recursive equilibria defined in 2.4, where the endowment of the lender is *constant and given by* $\tilde{y}^L$.

A final remark is that in both lemmas of this section, in principle, we allow for equilibria with negative consumption for the lender. Is it easy to see that the same results will hold for equilibria with only positive consumption, provided that endowments are large enough to sustain these at the equilibrium prices.

LEMMA E.1. *For any $\mathbb{E}(\tilde{y}^L)$, there exists an equilibrium in $\mathbb{E}(\widetilde{y}^L)$ such that: (a) the equilibrium price mapping is given by q^*; (b) the borrower equilibrium policy functions are given by B^* and δ^*; (c) the lenders' value functions are given by $W_i^* - \frac{y^L - \widetilde{y}^L}{\gamma}$, and consumption is given by $c^{*,L} - (y^L - \widetilde{y}^L)$.*

Proof of Lemma E.1. We divide the proof into several steps. We first show that, if the prices and default strategies are given by q^* and δ^*, then lenders' optimal debt policy is given by b^* and

$$\widetilde{W}_i(\cdot) = W_i^*(\cdot) - \frac{x}{1-\gamma}, \tag{15}$$

where $x \equiv y^L - \widetilde{y}^L$. We then show that the equilibrium price schedule is in fact q^*, given that default strategies are δ^*. Finally, we show that given the fact that equilibrium price schedule is in fact q^*, then default strategies and the borrower's debt policy functions are δ^* and B^*, respectively. This last part is trivial, since if the equilibrium price schedules are the same (between the equilibrium in $\mathbb{E}(\tilde{y}^L)$ and the one defined in 2.4), they will clearly support the same choices of debt and thus of default strategies for the borrower. We therefore show only the first two claims.

STEP 1. Suppose the equilibrium price mapping and default strategies are the same in both recursive equilibria (the one with y^L and the one with $\widetilde{y}^L$). In the economy with endowment $\widetilde{y}^L$ the lenders' value function when the borrower repays is given by

$$\widetilde{W}_R(y_t, B_t, b_t) = \max_{b' \in \mathbb{B}} \Big\{ \tilde{y}^L + q^*(y_t, B^*(y, B_t))b' - b_t \\ -\gamma\theta \log \int_{\mathbb{Y}} \exp\left\{ -\frac{\widetilde{W}(y_{t+1}, B^*(y_t, B_t), b')}{\theta} \right\} f(y_{t+1}|y_t) dy_{t+1} \Bigg\}. \tag{16}$$

Substituting our guess in equation (15) for the continuation value in the expression (16) leads to:

$$\begin{aligned}\widetilde{W}_R(y_t, B_t, b_t) =& \max_{b' \in \mathbb{B}} \Bigg\{ \tilde{y}^L + q^*(y_t, B^*(y_t, B_t))b' - b_t - \gamma \frac{x}{1-\gamma} \\ & -\gamma\theta \log \int_{\mathbb{Y}} \exp\left\{ -\frac{W^*(y_{t+1}, B^*(y_t, B_t), b')}{\theta} \right\} f(y_{t+1}|y) dy_{t+1} \Bigg\} \\ =& \max_{b' \in \mathbb{B}} \Bigg\{ \tilde{y}^L + q^*(y, B^*(y_t, B_t))b' - b_t - (\gamma - 1)\frac{x}{1-\gamma} \\ & -\gamma\theta \log \int_{\mathbb{Y}} \exp\left\{ -\frac{W^*(y_{t+1}, B^*(y_t, B_t), b')}{\theta} \right\} f(y_{t+1}|y_t) dy_{t+1} \Bigg\} - \frac{x}{1-\gamma} \\ =& \max_{b' \in \mathbb{B}} \Big\{ y^L + q^*(y_t, B^*(y, B_t))b' - b_t \\ & -\gamma\theta \log \int_{\mathbb{Y}} \exp\left\{ -\frac{W^*(y_{t+1}, B^*(y_t, B_t), b')}{\theta} \right\} f(y_{t+1}|y_t) dy_{t+1} \Bigg\} - \frac{x}{1-\gamma}.\end{aligned}$$

Summing $\frac{x}{1-\gamma}$ on both sides, implies that

$$W^*_R(y_t, B_t, b_t) = \max_{b' \in \mathbb{B}} \Big\{ y^L + q^*(y_t, B^*(y_t, B_t))b' - b_t \\ -\gamma\theta \log \int_{\mathbb{Y}} \exp\left\{ -\frac{W^*(y_{t+1}, B^*(y_t, B_t), b')}{\theta} \right\} f(y_{t+1}|y_t) dy_{t+1} \Bigg\}.$$

This verifies our guess for the lenders' value function when the borrower honors his debt contracts. Similar arguments can be applied to derive the value function in autarky $\widetilde{W}_A(y)$. Moreover, the optimal choices of debt coincide in the two recursive equilibria.

STEP 2. We now verify that equilibrium price schedules in fact coincide in the two recursive equilibria (the one with y^L and the one with $\widetilde{y}^L$), given the same debt and default strategies for the borrower. This is analogous to showing that the equilibrium probability distortions m^* are the same in the two equilibria, given the same debt and default strategies.

Given any $B' \in \mathbb{B}$, in equilibrium, the probability distortion to the conditional density for next-period borrower's endowment when the borrower repays in the current state, is given by the twisting formula

$$m_i^*(y_t, y_{t+1}, B') = \frac{\exp\left\{-\frac{W_i^*(y_{t+1}, B', B')}{\theta}\right\}}{\int_{\mathbb{Y}} \exp\left\{-\frac{W_i^*(y_{t+1}, B', B')}{\theta}\right\} f(y_{t+1}|y_t) dy_{t+1}}.$$

Since $\widetilde{W}_i(\cdot) = W_i^*(\cdot) - \frac{x}{1-\gamma}$, and x is constant. It is straightforward to show that

$$m_i^*(y_t, y_{t+1}, B') \quad = \quad \frac{\exp\left\{-\frac{\widetilde{W}_i(y_{t+1}, B', B') + \frac{x}{1-\gamma}}{\theta}\right\}}{\int_{\mathbb{Y}} \exp\left\{-\frac{\widetilde{W}_i(y_{t+1}, B', B') + \frac{x}{1-\gamma}}{\theta}\right\} f(y_{t+1}|y_t) dy_{t+1}}.$$

Hence, the same probability distortions are optimal from the minimizing agent's perspective in the economy with lenders' endowment $\widetilde{y}^L$. □

The next lemma is a formal reformulation of Lemma 3.2. In order to do this, we need to introduce some additional notation.

Let a be the amount of savings the agent has in the risk-free asset, and let A be the aggregate amount of savings in the risk-free asset. Let $(y, B, b, A, a) \mapsto a(y, B, b, A, a)$ be the optimal policy correspondence given a state (y, B, b, A, a). Abusing notation, we also denote $a(y, B, b, A, a)$ as a selection of the policy function. Let $(y, B, b, A, a) \mapsto \widetilde{W}_R(y, B, b, A, a)$ be the value function of a lender when there is access to financial markets, and let $(y, A, a) \mapsto \widetilde{W}_A(y, A, a)$ be the one when the economy is in financial autarky.

A recursive equilibrium *with the risk-free asset and the endowment of the lender constant and given by* y^L is defined analogously to our definition of recursive equilibrium (see definition 2.4), that is, $\{c^*, c^{*,L}, B^*, b^*, m^*, \delta^*, a^*, A^*\}$, the collection of value functions $\{V_R^*, V_A^*, \widetilde{W}_R^*, \widetilde{W}_A^*\}$, the perceived laws of motion for the borrower's bond holdings and aggregate savings (Γ, Φ), and the price schedule q and a gross risk-free rate R. The only additional condition is the market-clearing one for the new asset:

$$\bar{A} = a^*(y, B, B, A, A), \ \forall y, B, A,$$

and

$$\Phi(y, B, A) = A^*(y, B, A) = a^*(y, B, B, A, A), \ \forall y, B, A.$$

For any $\tilde{y}^L$ we denote the set of recursive equilibria with the risk-free asset and the endowment of the lender constant and given by $\tilde{y}^L$, as $\mathbb{E}^{rf}(\tilde{y}^L)$.

Lemma E.2. *For any* $\mathbb{E}^{rf}(\tilde{y}^L)$*, there exists an equilibrium in* $\mathbb{E}^{rf}(\tilde{y}^L)$*, such that: (a) the equilibrium price mapping is given by* q^**; (b) the borrower equilibrium policy functions are given by* B^* *and* δ^**; (c) the lenders' value functions are given by* $(y, B, b, A, a) \mapsto W_i^*(y, B, b) + a$ *and debt policy functions are given by* b^**.*

Proof of Lemma E.2. The proof is very similar to that of Lemma E.1, so we consider only the main steps.

STEP 1. First, note that in equilibrium $\gamma^{-1} = R \equiv 1 + r_f$. If this does not hold, the foreign lenders could achieve unbounded consumption, which cannot happen in equilibrium.

Given $R^{-1} = \gamma$, we now show that $\tilde{W}_j(y, B, b, A, a) = W_j(y, B, b) + a$ and $b(y, B, b, a) = b^*(y, B, b)$ for $j = \{R, A\}$, given that the equilibrium prices, borrower's debt policy function, and default strategies coincide for the two equilibria (the one with risk-free asset and the one without).

It follows that

$$\begin{aligned}\tilde{W}_R(y_t, B_t, b_t, A_t, a_t) = & \, y^L + q(y_t, B^*(y_t, B_t))\, b(y, B_t, b_t, A_t, a_t) - b_t - R^{-1} a(y_t, B_t, b_t, A_t, a_t) + a_t \\ & + \gamma T^\theta[\tilde{W}(\cdot, B^*(y_t, B_t), b(y_t, B_t, b_t, A_t, a_t), a(y_t, B_t, b_t, A_t, a_t))](y_t).\end{aligned}$$

We first show that, given $b(y, B, b, A, a) = b^*(y, B, b)$, $W^*(y, B, b) + a$ is a fixed point of the previous equation. For this note that (we omit the inputs on q to ease the notational burden)

$$\begin{aligned}\tilde{W}_R(y_t, B_t, b_t, A_t, a_t) = & \, y^L + q^*(\cdot) b^*(y_t, B_t, b_t) - b_t - R^{-1} a(y_t, B_t, b_t, A_t, a_t) + a_t + \\ & \gamma T^\theta[\tilde{W}(\cdot, B^*(y_t, B_t), b^*(y, B_t, b_t), a(y_t, B_t, b_t, A_t, a_t))](y_t) \\ = & \, y^L + q^*(\cdot) b^*(y_t, B_t, b_t) - b_t - R^{-1} a(y_t, B_t, b_t, A_t, a_t) + a_t + \\ & \gamma T^\theta[W^*(\cdot, B^*(y_t, B_t), b^*(y_t, B_t, b_t)) + a(y_t, B_t, b_t, A_t, a_t)](y_t) \\ = & \, y^L + q^*(\cdot) b^*(y_t, B_t, b_t) - b_t - R^{-1} a(y_t, B_t, b_t, A_t, a_t) + a_t + \\ & \gamma T^\theta[W^*(\cdot, B^*(y_t, B_t), b^*(y_t, B_t, b_t))](y_t) + \gamma a(y_t, B_t, b_t, A_t, a_t) \\ = & \left\{y^L + q^*(\cdot) b^*(y_t, B_t, b_t) - b_t\right\} + \gamma T^\theta[W^*(\cdot, B^*(y_t, B_t), b^*(y_t, B_t, b_t))](y_t) \Big\} + a_t \\ = & W_R^*(y_t, B_t, b_t) + a_t.\end{aligned}$$

A similar result holds for $\tilde{W}_A$. Hence, since $\tilde{W}$ is a convex combination of $\tilde{W}_A$ and $\tilde{W}_R$, the result follows.

We now show that, given $\tilde{W}_R(y, B, b, A, a) = W_R^*(y, B, b) + a$, $b(y, B, b, A, a) = b^*(y, B, b)$ is optimal. This is easy to see, since if $W(y, B, b, A, a) = W^*(y, B, b) + a$, then a enters as a constant, and thus it does not affect the argument that maximizes the Bellman problem.

STEP 2. We now show that the equilibrium price function q is the same as that with

$a \equiv 0$. By step 1, $\tilde{W}_R(y, B, b, A, a) = W_R^*(y, B, b) + a$, hence

$$\begin{aligned}
\tilde{q}(y_t, B') &\equiv \int_{D(B')^C} \frac{\exp\left\{-\frac{\tilde{W}(y_{t+1}, B', B', A', A')}{\theta}\right\}}{\int \exp\left\{-\frac{\tilde{W}(y_{t+1}, B', B', A', A')}{\theta}\right\} f(y_{t+1}|y_t) dy_{t+1}} f(y_{t+1}|y_t) dy_{t+1} \\
&= \int_{D(B')^C} \frac{\exp\left\{-\frac{W^*(y_{t+1}, B', B') + A'}{\theta}\right\}}{\int \exp\left\{-\frac{W^*(y_{t+1}, B', B') + A'}{\theta}\right\} f(y_{t+1}|y_t) dy_{t+1}} f(y_{t+1}|y_t) dy_{t+1} \\
&= \int_{D(B')^C} \exp\left\{-\frac{A'}{\theta}\right\} \frac{\exp\left\{-\frac{W^*(y_{t+1}, B', B')}{\theta}\right\}}{\left(\int \exp\left\{-\frac{W^*(y_{t+1}, B', B')}{\theta}\right\} f(y_{t+1}|y_t) dy_{t+1}\right) \exp\left\{-\frac{A'}{\theta}\right\}} f(y_{t+1}|y_t) dy_{t+1} \\
&= \int_{D(B')^C} \frac{\exp\left\{-\frac{W^*(y_{t+1}, B', B')}{\theta}\right\}}{\left(\int \exp\left\{-\frac{W^*(y_{t+1}, B', B')}{\theta}\right\} f(y_{t+1}|y_t) dy_{t+1}\right)} f(y_{t+1}|y_t) dy_{t+1},
\end{aligned}$$

where $A' = a(y, B, B, A, A)$, and the third line follows from the fact that A' is constant with respect to y'. The last line is the same expression as in the case for $A \equiv 0$, and thus the desired result follows.

The fact that the borrower's allocations remain unchanged follows immediately from this result and from the fact that they do not depend on choices of the individual lenders. □

F Economy with Incomplete Information

We consider a slightly different version of the borrower's problem, and thus of the whole economy, studied in the text. We now assume that in every period t, after the state is realized, the borrower receives an i.i.d.- preference shock ν, drawn from $P_\nu(\cdot|s)$, before choosing its action. The preference shock ν is known to the borrower but not to the lender. The period payoff of the borrower is given by $u + \nu \times (1 - \delta)$. We refer to this new economy with private information as the perturbed economy.

Throughout this section we changed the notation used in the text to a more succinct one (albeit less standard). This new notation follows closely that in Doraszelski and Escobar (2010) and facilitates the proof of the theoretical results in this section.

Let $s \equiv (B, y, \phi) \in \mathbb{S} \equiv \mathbb{B} \times \mathbb{Y} \times \{0, 1\}$ be the aggregate state, where B is the government debt, y is the endowment and ϕ is an indicator variable such that $\phi = 0$ states that the economy is in financial access, and $\phi = 1$ states that the economy is in financial autarky. The law of motion for ϕ is given by

$$\Pr\{\phi' = 0 \mid \phi = 1\} = \pi$$

and

$$\Pr\{\phi' = 0 \mid \phi = 0\} = \sigma(1|s) + \sigma(0|s)\pi,$$

where $\sigma(\delta|s)$ is the probability of the borrower to choose δ, given s (see below).

Recall that $\delta \in \{0, 1\}$ is the action of default of the government; $\delta = 1$ means no default and $\delta = 0$ means default. The possible actions of default are state dependent in the following sense $A(s) = \{0, 1\}$ if $\phi = 0$ and $A(s) = \{0\}$ if $\phi = 1$. That is, if the economy is in financial autarky, then the government does not have a choice and ought to remain in financial autarky. We use $\mathbf{B}$ to denote the government debt policy function, that is, $\mathbf{B}\colon \mathbb{S} \times \{0, 1\} \to \mathbb{R}$ such that $\mathbf{B}(s; q)$ is the choice of new bond holdings, given the state s, a choice of default $\delta = 1$ (there is no need to define $\mathbf{B}$ when $\delta = 0$ since the economy is in financial autarky) and a belief about the pricing function.

Let $s_L \equiv (b, s) \in \mathbb{B} \times \mathbb{S}$ be the state of the lender. We use $\mathbf{b}$ to denote the debt policy function for the lenders, that is, $\mathbf{b}(s_L; \sigma, \mathbf{B}, q)$ is the the choice of lender's debt for "tomorrow," given the state s_L and beliefs about $\sigma, \mathbf{B}, q$.

Our goal is to show that when $P_\nu(\cdot|s)$ converges to a degenerate measure at zero (in a certain sense to be specified below), the solution of the perturbed economy, if it converges, does so to the solution of the economy without preference shock. Hence, we consider a sequence of cdfs $(F^n_\nu(\cdot|s))_{n\in\mathbb{N}}$ that has the following property

Assumption F.1. *For any $s \in \mathbb{S}$ and $n \in \mathbb{N}$, (i) if $\phi = 1$, then $P^n_\nu(v|s) = \delta_0(v)$ and if $\phi = 0$, then $F^n_\nu(\cdot|s)$ is absolutely continuous with respect to the Lebesgue measure, and*

$$\lim_{n\to\infty} \sup_{s\in\mathbb{S}} \frac{1}{P^n_\nu(A^n|s)} \int_{A^n} \nu F^n_\nu(d\nu|s) = 0,$$

for any sequence of Borel measurable sets A^n.

This assumption is analogous to that in Doraszelski and Escobar (2010). The only caveat is that, due to the characteristics of our problem, we impose that $\nu \equiv 0$ whenever the economy is in financial autarky. It is only when the government *actually chooses to default* that the shock is present. This particular choice stems from the fact that our interest in the shock is merely as a numerical device to "smooth" the default decision.

We use $\{(W^n, V^n), (\mathbf{b}^n, \mathbf{B}^n), \sigma^n, q^n\}$, to denote quantities corresponding to the perturbed economy with preference shock distribution F^n_ν, and we include a superscript $*$ to denote equilibrium quantities.

F.1 Borrower's Problem

We first present the transition probabilities of the state for this new formulation and then present the Bellman equation for the borrower.

F.1.1 Transition state probabilities

Let $C \equiv C_B \times C_Y \times C_\phi$, where C_B and C_Y are Borel measurable subsets of $\mathbb{Y}$ and $\mathbb{B}$, and $C_\phi \in 2^{\{0,1\}}$. For almost any $\bar{s} \equiv (\bar{B}, \bar{y}, \bar{\phi}) \in \mathbb{S}$ we compute the conditional probability of C, given $\bar{s}$ and given the choices δ and $\tilde{B}$, that is, $Q\left(s' \in C \mid s = \bar{s}; \sigma, \tilde{B}\right)$. For the case $\bar{\phi} = 0$,

$$\begin{aligned} Q\left(s' \in C \mid s = \bar{s}; \sigma, \tilde{B}\right) =& 1\{\tilde{B} \in C_B\}(1\{0 \in C_\phi\}\sigma(0|\bar{s}) + [\pi 1\{0 \in C_\phi\} + (1-\pi)1\{1 \in C_\phi\}]\sigma(1|\bar{s})) \\ & \times \int_{y'\in C_Y} f(y'|\bar{y})dy'. \end{aligned}$$

For the case $\bar{\phi} = 1$,

$$Q\left(s' \in C \mid s = \bar{s}; \sigma, \tilde{B}\right) = 1\{\tilde{B} \in C_B\}(1\{0 \in C_\phi\}\pi + 1\{1 \in C_\phi\}(1-\pi)) \int_{y' \in C_Y} f(y'|\bar{y})dy'.$$

F.1.2 Recursive Formulation

The borrower's Bellman equation can be written succinctly as

$$V^n(s;q) = \int_{\mathbb{R}} \left(\max_{\delta \in A(s)} \mathcal{V}^n(\delta, s; q) + \nu(1-\delta) \right) F_\nu^n(d\nu|s), \tag{17}$$

where $\nu(1-\delta)$ implies that the preference shock occurs only if the borrower is actually choosing default, and not if it is already in financial autarky; also

$$\mathcal{V}^n(\delta, s; q) \equiv U(s, \delta, \mathbf{B}^n(s;q); q) + \beta \int_{\mathbb{S}} V^n(s';q) Q(ds' \mid s; \delta, \mathbf{B}^n(s;q)),$$

and,

$$\mathbf{B}^n(s;q) \in \arg\max_{B' \in \mathbb{B}(\delta)} \left\{ U(s, \delta, B'; q) + \beta \int_{\mathbb{S}} V^n(s';q) Q(ds' \mid s; \delta, B') \right\},$$

where $\mathbb{B}(\delta) \equiv \mathbb{B}\delta + (1-\delta)\{0\}$. Finally for all $s \in \mathbb{S}$,

$$U(s, \delta, B'; q) \equiv (1-\delta)u(y - \phi(y)) + \delta u(y - q(y, B')B' + B).$$

Let $\delta(s, \nu)$ be the best response of the borrower in state s, and shock ν, that is,

$$\delta^n(s, \nu) = \begin{cases} 1 & if\ \mathcal{V}^n(1, s; q) > \mathcal{V}^n(0, s; q) + \nu \\ 0 & if\ \mathcal{V}^n(1, s; q) < \mathcal{V}^n(0, s; q) + \nu. \end{cases}$$

We note that, since ν is assumed to have an absolutely continuous distribution; the case $\mathcal{V}^n(1, s; q) = \mathcal{V}^n(0, s; q) + \nu$ occurs with probability zero.

Note that

$$\sigma^n(a|s) \equiv \int_{\nu:\ \delta^n(s,\nu) = a} F_\nu^n(d\nu|s).$$

We let $U(s, \sigma, B'; q) \equiv \sigma(1|s)U(s, 1, B'; q) + \sigma(0|s)U(s, 0, B'; q)$. The next lemma characterizes the properties of V^n.

LEMMA F.1. *Suppose Assumption F.2 holds and $q(y, \cdot)$ is continuous. Then (1) For each $B' \in \mathbb{B}$, $U(\cdot, \sigma, B'; q^*)$ is a uniformly bounded and continuous function and (2) $V^n(\cdot; q)$ is a uniformly bounded and continuous function.*

Proof. The proof follows from standard arguments; see Stokey and Robert E. Lucas (1989). □

For example, if F_ν^n is given by the logistic distribution, that is, $\nu \mapsto F_\nu^n(\nu) = \frac{1}{1+\exp\{-h_n^{-1}\nu\}}$ (h_n controls the variance of F_ν^n), equation (17) has a closed-form solution, given by

$$\begin{aligned}
V^n(s;q) =& \int_{\mathbb{R}} \max\left\{0, \mathcal{V}^n(1,s;q) - \mathcal{V}^n(0,s;q) + \nu\right\} F_\nu^n(d\nu) + \mathcal{V}^n(0,s;q) \\
=& \int_{-\Delta\mathcal{V}^n(s;q)}^{\infty} \{\Delta\mathcal{V}^n(s;q) + \nu\} F_\nu^n(d\nu) + \mathcal{V}^n(0,s;q) \\
=& -\nu(1 - F_\nu^n(\nu)) \mid_{\nu=0}^{\nu=\infty} + \int_0^\infty 1 - F_\nu^n(\nu - \Delta\mathcal{V}^n(s;q)) d\nu \\
=& h_n \log\left(1 + \exp\left\{h_n^{-1}\Delta\mathcal{V}^n(s;q)\right\}\right),
\end{aligned}$$

where $\Delta\mathcal{V}^n(s;q) = \mathcal{V}^n(1,s;q) - \mathcal{V}^n(0,s;q)$. The third line follows by integration by parts. The fourth line follows from the properties of the logistic distribution.

F.2 The Lender's Problem

For the sake of presentation, and given that we introduced new notation, we present the lender's problem. This part, as opposed to the borrower's, is just a reformulation of what we have in the text.

The conditional probability is given by $Q_L\left(\cdot \mid s_L = \bar{s}_L; \tilde{b}, \sigma, \mathbf{B}\right)$, where $\mathbf{B}$ and σ are beliefs of the lender about the policy functions of the government; the functional form is analogous to that of Q and thus, it will not be derived again.

Let

$$W(s_L;\sigma,\mathbf{B},q) = \left\{\mathbb{U}(s_L, \mathbf{b}(s_L;\sigma,\mathbf{B},q);\sigma,\mathbf{B},q) + \gamma T^\theta[W](s_L, \mathbf{b}(s_L;\sigma,\mathbf{B},q);\sigma,\mathbf{B},q)\right\},$$

where

$$\mathbf{b}(s_L;\sigma,\mathbf{B},q) \in \arg\max_{b'\in\mathbb{B}(\phi)} \left\{\mathbb{U}(s_L, b';\sigma,\mathbf{B},q) + \gamma T^\theta[W](s_L, b';\sigma,\mathbf{B},q)\right\},$$

where $\mathbb{B}(\phi) \equiv (1-\phi)\mathbb{B} + \phi\{0\}$. And for all $s_L \in \mathbb{S}_L$,

$$\mathbb{U}(s_L, b';\sigma,\mathbf{B},q) \equiv \sigma(0|d)y + \sigma(1|d)(y + q(y, \mathbf{B}(s;q))b' - b),$$

and T^θ is given by

$$W \mapsto T^\theta[W](s_L;\tilde{b},\sigma,\mathbf{B}) = -\theta \log\left(\int_{\mathbb{Y}} \exp\left\{-\frac{\mathcal{W}(y', s_L;\tilde{b},\sigma,\mathbf{B})}{\theta}\right\}\right),$$

where

$$\mathcal{W}(y', s_L;\tilde{b},\sigma,\mathbf{B}) = \int_{\mathbb{S}} W(s'_L) Q_L(ds'_L \mid y', s_L;\tilde{b},\sigma,\mathbf{B}),$$

(the measure used for integration is the measure Q_L conditional on y'; abusing notation we still denote it as Q_L).

The problem of the lender is not indexed by n directly; it will depend on n only indirectly through the lender's (correct) beliefs of the borrower's policy functions and prices. For convenience, however, we use $W^n(s_L; q^n)$ to denote $W(s_L; \sigma^n, \mathbf{B}^n, q^n)$.

The next lemma characterizes the properties of W.

LEMMA F.2. *Suppose $q(y, \cdot)$ is continuous. Then $W(\cdot; q)$ is a bounded and continuous function.*

Proof. The proof follows from standard arguments; see Stokey and Robert E. Lucas (1989). □

F.3 Convergence of the Economy with Incomplete Information

The definition of equilibrium in the economy with incomplete information is analogous to the one in the text and therefore, will be omitted. We denote the equilibrium of the perturbed economy as $\{(W^{*,n}, V^{*,n}), (\mathbf{b}^{*,n}, \mathbf{B}^{*,n}), \sigma^{*,n}, q^{*,n}\}$.

We note that, if q is an equilibrium price (given the other equilibrium quantities), then

$$q(s, B') = \gamma \int_{y' \,:\, \delta(B', y', 0)=1} m_R^*(y', y, B') f(y'|y) dy' \leq \gamma. \tag{18}$$

Hence $(s, B') \mapsto q(s, B') \in [0, \gamma]$.

The next assumption ensures that positive consumption for the borrower is always feasible.

ASSUMPTION F.2. *There exists a $C_1 > 0$, such that, for all $(y, B) \in \mathbb{Y} \times \mathbb{B}$,*

$$\sup_{B' \in \mathbb{B}} (y + B) - \gamma B' \geq C_1.$$

We now establish the main result of this section: as shocks vanishes, and if $(\mathbf{b}^{*,n}, \mathbf{B}^{*,n}, \sigma^{*,n}, q^{*,n})$ converge, then the equilibrium of the perturbed game converges to an equilibrium in economy without noise, that is, the economy described in the text.

LEMMA F.3. *Suppose Assumptions F.1 and F.2 hold, and suppose $q^{*,n}(y, \cdot)$ is continuous for each n. If $\lim_{n\to\infty} ||\sigma^{*,n} - \sigma^*||_{L^\infty(\{0,1\}\times\mathbb{S})} = 0$, $\lim_{n\to\infty} ||\mathbf{B}^{*,n} - \mathbf{B}^*||_{L^\infty(\mathbb{S})} = 0$ and $\lim_{n\to\infty} ||q^{*,n} - q^*||_{L^\infty(\mathbb{S})} = 0$, then $(\sigma^*, \mathbf{B}^*)$ are an equilibrium default strategy and debt policy function, and q^* is a continuous equilibrium price function of the economy without noise.*[62]

Proof. We divide the proof into several steps. Throughout the proof *supp*$(\sigma(\cdot|s))$ is the set of all d such that $\sigma(d|s) > 0$.

STEP 1. We show that if $\delta \in$ *supp*$(\sigma(\cdot|s))$, then $\delta \in$ *argmax* $(\mathcal{V}^{*,n}(\delta, s; q))$. Since $\lim_{n\to\infty} ||\sigma^{*,n} - \sigma^*||_{L^\infty(\{0,1\}\times\mathbb{S})} = 0$, there exists an N, such that *supp*$(\sigma^*(\cdot|s)) \subseteq$ *supp*$(\sigma^{*,n}(\cdot|s))$

[62]The norm $||\cdot||_{L^\infty(X)}$ is the standard supremum norm over functions that map X to $\mathbb{R}$.

for all $n \geq N$. For any $\delta \in supp(\sigma^{*,n}(\cdot|s)) = supp(g^n(\cdot|s))$, since F^n_ν is absolutely continuous with respect to the Lebesgue measure, there exists a set $D^n_s \subseteq \mathbb{R}$ (of positive measure) such that

$$\mathcal{V}^{*,n}(\delta, s; q) - \mathcal{V}^{*,n}(\delta', s; q) > \nu(1-\delta') - \nu(1-\delta)$$

for all $\delta' \neq \delta$ and all $\nu \in D^n_s$. Integrating at both sides it follows that

$$\mathcal{V}^{*,n}(\delta, s; q) - \mathcal{V}^{*,n}(\delta', s; q) > \frac{1}{P^n_\nu(D^n_s)} \int_{\mathbb{R}} \{\nu(1-\delta)' - \nu(1-\delta)\} F^n_\nu(d\nu|s).$$

As $n \to \infty$, by Assumption F.1, the RHS converges to zero, and by Lemma F.4, the LHS converges to $\mathcal{V}^*(\delta, s; q) - \mathcal{V}^*(\delta', s; q)$, and the result thus follows.

Step 2. We show that, for any s and d, $\mathbf{B}^*(s)$ is optimal for the borrower, given σ^* and q^*. It suffices to show this for $\delta = 1$ (for $\delta = 0$ there is no choice of debt). We note that since $q^{*,n}$ is continuous and converges uniformly to q^* over a compact set, q^* is also continuous.

We know that

$$U(s, 0, \mathbf{B}^{*,n}(s; q^{*,n}); q^{*,n}) + \beta \int_{\mathbb{S}} V^{*,n}(s'; q^{*,n}) Q(ds' \mid s; 0, \mathbf{B}^{*,n}) \geq U(s, 0, B'; q^{*,n}) \tag{19}$$

$$+ \beta \int_{\mathbb{S}} V^{*,n}(s'; q^{*,n}) Q(ds' \mid s; 0, B') \tag{20}$$

for all $B' \in \mathbb{B}$.

Note that

$$\begin{aligned}
\sup_{B'} \left| U(s, 0, B'; q^{*,n}) - U(s, 0, B'; q^*) \right| =& \sup_{B'} \left| u(y + B - q^{*,n}(y, B')B') - u(y + B - q^*(y, B')B') \right| \\
\leq& \sup_{B'} \left| \frac{du(y + B - \bar{q}^{*,n}(y, B')B')}{dc} \right| \left| B'\{q^{*,n}(y, B') - q^*(y, B')\} \right| \\
=& O(||q^{*,n} - q^*||_{L^\infty(\mathbb{S})}) = o(1),
\end{aligned} \tag{21}$$

where $\bar{q}^{*,n}(y, B')$ is a mid-point between $q^{*,n}(y, B')$ and $q^*(y, B')$. The second line follows from the differentiability of u; the third line follows from the fact that $\mathbb{B}$ is compact (and hence bounded) and $\sup_{B'}\{y + B - \bar{q}^{*,n}(y, B')B'\} \geq C_1$ (by Assumption F.2) and $c \mapsto \frac{du(c)}{dc}$ is bounded at C_1. Also, by Lemma F.4, we know that, for any s,

$$\sup_{B'} \left| \int_{\mathbb{S}} V^{*,n}(s'; q^{*,n}) Q(ds' \mid s; 0, B') - \int_{\mathbb{S}} V^*(s'; q^*) Q(ds' \mid s; 0, B') \right| = o(1). \tag{22}$$

Hence (we omit the inputs in the policy function to ease the notational burden)

$$\begin{aligned}
&U(s,0,\mathbf{B}^*;q^*)+\beta\int_{\mathbb{S}}V^*(s';q^*)Q(ds'\mid s;0,\mathbf{B}^*)\\
&\geq U(s,0,\mathbf{B}^{*,n};q^{*,n})+\beta\int_{\mathbb{S}}V^{*,n}(s';q^{*,n})Q(ds'\mid s;0,\mathbf{B}^{*,n})-A_n(s)\\
&\geq U(s,0,B';q^{*,n})+\beta\int_{\mathbb{S}}V^{*,n}(s';q^{*,n})Q(ds'\mid s;0,B')-A_n(s)\\
&\geq U(s,0,B';q^*)+\beta\int_{\mathbb{S}}V^*(s';q^*)Q(ds'\mid s;0,B')-2A_n(s),
\end{aligned}$$

where

$$\begin{aligned}
A_n(s)\equiv\sup_{B'}\Bigg|&U(s,0,B';q^{*,n})+\beta\int_{\mathbb{S}}V^{*,n}(s';q^{*,n})Q(ds'\mid s;0,B')\\
&-\left\{U(s,0,B';q^*)+\beta\int_{\mathbb{S}}V^*(s';q^*)Q(ds'\mid s;0,B')\right\}\Bigg|,
\end{aligned}$$

and the second inequality follows from equation (19). This equation holds for any n, in particular as $n\to\infty$. By equation (22), $A_n(s)=o(1)$, and the result thus follows.

STEP 3. We show that, for any s, $B'\mapsto q^*(s,B')$ is an equilibrium price schedule that supports σ^* and $\mathbf{B}^*$. For this, it suffices to verify that

$$q^*(s,B')=\gamma\int_{\mathbb{Y}}\sigma(1|B',y',0)\frac{\exp\{-\theta^{-1}W^*(B',B',y',0;q^*)\}}{\int_{\mathbb{Y}}\exp\{-\theta^{-1}W^*(B',B',y',0;q^*)\}}f(y'|y)dy'.$$

To establish this, we note that for all (s,B'),

$$\begin{aligned}
&\left|q^*(s,B')-\gamma\int_{\mathbb{Y}}\sigma^*(1|B',y',0)\frac{\exp\{-\theta^{-1}W^*(B',B',y',0;q^*)\}}{\int_{\mathbb{Y}}\exp\{-\theta^{-1}W^*(B',B',y',0;q^*)\}}f(y'|y)dy'\right|\\
&\leq|q^*(s,B')-q^{*,n}(s,B')|\\
&\quad+\Bigg|\gamma\int_{\mathbb{Y}}\Bigg(\sigma^{*,n}(1|B',y',0)\frac{\exp\{-\theta^{-1}W^{*,n}(B',B',y',0;q^{*,n})\}}{\int_{\mathbb{Y}}\exp\{-\theta^{-1}W^{*,n}(B',B',y',0;q^{*,n})\}}\\
&\quad-\sigma^*(1|B',y',0)\frac{\exp\{-\theta^{-1}W^*(B',B',y',0;q^*)\}}{\int_{\mathbb{Y}}\exp\{-\theta^{-1}W^*(B',B',y',0;q^*)\}}\Bigg)f(y'|y)dy'\Bigg|.
\end{aligned}$$

The first term on the RHS vanishes as $n\to\infty$ by assumption. Since $\lim_{n\to\infty}||\sigma^{*,n}-\sigma^*||_{L^\infty(\{0,1\}\times\mathbb{S})}=0$ and $W\mapsto\frac{\exp\{-\theta^{-1}W\}}{\int_{\mathbb{Y}}\exp\{-\theta^{-1}W\}}$ is continuous (under the $||\cdot||_{L^\infty(\mathbb{S})}$ norm), it suffices to show that $||W^{*,n}(\cdot;q^{*,n})-W^*(\cdot;q^*)||_{L^\infty(\mathbb{S})}=o(1)$. This follows from Lemma F.4 and similar calculations to those done in equations (21) and (22). □

In order to establish Lemma F.4 below, it is convenient to introduce some notation. Let

$\mathbb{L}_{\sigma,B}\colon L^\infty(\mathbb{S}) \to L^\infty(\mathbb{S})$, such that

$$\mathbb{L}_{\sigma,\mathbf{B}}[v](s) \equiv \int_{\mathbb{S}} v(s')Q(ds' \mid s;\sigma,\mathbf{B}(s;q)).$$

It is easy to see that $\mathbb{L}_{\sigma,B}$ is a linear operator. Let $\mathcal{L}(L^\infty(\mathbb{S}))$ be the class of linear operators that map $L^\infty(\mathbb{S})$ onto $L^\infty(\mathbb{S})$. Endowed with the operator norm, that is, $||L_{\sigma,B}|| \equiv \sup_{v\in L^\infty(\mathbb{S})} \frac{||\mathbb{L}_{\sigma,B}[v]||_{L^\infty(\mathbb{S})}}{||v||_{L^\infty(\mathbb{S})}}$, the class $\mathcal{L}(L^\infty(\mathbb{S}))$ is a Banach algebra, see Lax (2002). This implies that algebraic properties enjoyed by algebras, such as multiplication of elements, can be applied to $\mathbb{L}_{\sigma,B}$. In particular, following analogous steps to those in Doraszelski and Escobar (2010) p. 381, it follows that

$$\begin{aligned} V^n(s;q) =& U(s,\sigma,\mathbf{B}(s;q);q) + \sigma(0|s)\int_{\nu\colon\, \delta^n(s,\nu)=0} \nu F^n_\nu(d\nu|s) + \beta\int_{\mathbb{S}} V^n(s';q)Q(ds' \mid s;\sigma,\mathbf{B}(s;q)) \\ \equiv& U(s,\sigma,\mathbf{B}(s;q);q) + e^n(s;\sigma) + \beta\int_{\mathbb{S}} V^n(s';q)Q(ds' \mid s;\sigma,\mathbf{B}(s;q)). \end{aligned}$$

can be cast as

$$V^n(s;q) = \left(\sum_{t=0}^{\infty} \beta^t \mathbb{L}^t_{\sigma,\mathbf{B}}\right)[U(\cdot,\sigma,\mathbf{B}(s,\sigma;q);q) + e^n(\cdot;\sigma)](s),$$

where $\mathbb{L}^t_{\sigma,\mathbf{B}}$ is the composition of the operator applied t times. Equivalently (see Lemma F.5)

$$V^n(s;q) = (I - \beta\mathbb{L}_{\sigma,\mathbf{B}})^{-1}[U(\cdot,\sigma,\mathbf{B}(s;q);q) + e^n(\cdot;\sigma)](s).$$

In particular,

$$V^{*,n}(s;q^{*,n}) = (I - \beta\mathbb{L}_{\sigma^{*,n},\mathbf{B}^{*,n}})^{-1}[U(\cdot,\sigma^{*,n},\mathbf{B}^{*,n}(s;q^{*,n});q^{*,n}) + e^n(\cdot;\sigma^{*,n})](s). \tag{23}$$

And,

$$V^*(s;q^*) = (I - \beta\mathbb{L}_{\sigma^*,\mathbf{B}^*})^{-1}[U(\cdot,\sigma^*,\mathbf{B}^*(s;q^*);q^*)](s). \tag{24}$$

Let $\mathbb{C}(\mathbb{S})$ be the class of continuous functions in $L^\infty(\mathbb{S})$.

Lemma F.4. *Suppose Assumption F.2 holds, $q^*(y,\cdot)$ is continuous, and suppose* $\lim_{n\to\infty} ||\sigma^{*,n} - \sigma^*||_{L^\infty(\{0,1\}\times\mathbb{S})} = 0$, $\lim_{n\to\infty} ||\mathbf{B}^{*,n} - \mathbf{B}^*||_{L^\infty(\mathbb{S})} = 0$ *and* $\lim_{n\to\infty} ||q^{*,n} - q^*||_{L^\infty(\mathbb{S})} = 0$. *Then, for any $s \in \mathbb{S}$,*

$$\lim_{n\to\infty}\sup_{B'}\left|\int_{\mathbb{S}} V^{*,n}(s';q^{*,n})Q(ds' \mid s;0,B') - \int_{\mathbb{S}} V^*(s';q^*)Q(ds' \mid s;0,B')\right| = o(1). \tag{25}$$

And

$$\lim_{n\to\infty}\sup_{B'}\left|\int_{\mathbb{S}} W^{*,n}(s';q^{*,n})Q_L(ds' \mid s;B',0,B') - \int_{\mathbb{S}} W^{*}(s';q^{*})Q_L(ds' \mid s;B',0,B')\right| = o(1). \tag{26}$$

Proof of Lemma F.4. The proof for equation (26) is completely analogous to that of equation (25); hence we only show the latter.

We note that

$$\int_{\mathbb{S}} V^{*,n}(s';q^{*,n})Q(ds' \mid s;0,B') = \int_{\mathbb{S}} V^{*,n}(B',y',0;q^{*,n})f(y'|y)dy'$$

(and the same for V^*). Hence, since $V^{*,n}$ (and V^*) belong to $L^\infty(\mathbb{S})$ (see Lemma F.1), by the Dominated Convergence Theorem, it suffices to show that

$$\lim_{n\to\infty}\sup_{B'}|V^{*,n}(B',y',0;q^{*,n}) - V^{*}(B',y',0;q^{*})| = 0,$$

pointwise on $y' \in \mathbb{Y}$. By equations (23)-(24), we can cast $V^{*,n}$ and V^* in terms of $(I-\beta\mathbb{L}_{\sigma,\mathbf{B}})^{-1}$ and thus, it suffices to show that

$$\lim_{n\to\infty}\sup_{B}|\mathbb{M}_{\sigma^{*,n},\mathbf{B}^{*,n}}[e^n(\cdot;\sigma^{*,n})](s)| = 0, \tag{27}$$

$$\lim_{n\to\infty}\sup_{B}|\{\mathbb{M}_{\sigma^{*,n},\mathbf{B}^{*,n}} - \mathbb{M}_{\sigma^{*},\mathbf{B}^{*}}\}[U(\cdot,\sigma^*,\mathbf{B}^*(s;q^*);q^*)](s)| = 0, \tag{28}$$

and

$$\lim_{n\to\infty}\sup_{B}|\mathbb{M}_{\sigma^{*},\mathbf{B}^{*}}[U(\cdot,\sigma^*,\mathbf{B}^*(s;q^{n,*});q^{n,*}) - U(\cdot,\sigma^*,\mathbf{B}^*(s;q^*);q^*)](s)| = 0, \tag{29}$$

where $\mathbb{M}_{\sigma,\mathbf{B}} \equiv (I-\beta\mathbb{L}_{\sigma,\mathbf{B}})^{-1}$.

Regarding equation (27), note that

$$\sup_{B}|\mathbb{M}_{\sigma^{*,n},\mathbf{B}^{*,n}}[e^n(\cdot;\sigma^{*,n})](s)| \leq ||\mathbb{M}_{\sigma^{*,n},\mathbf{B}^{*,n}}|| \times ||e^n(\cdot;\sigma^{*,n})||_{L^\infty(\mathbb{S})}.$$

The operator norm of $\mathbb{M}_{\sigma^{*,n},\mathbf{B}^{*,n}}$, $||\mathbb{M}_{\sigma^{*,n},\mathbf{B}^{*,n}}|| \leq \sum_t \beta^t ||\mathbb{L}_{\sigma^{*,n},\mathbf{B}^{*,n}}||^t \leq (1-\beta)^{-1}$. Hence, it suffices to show that $||e^n(\cdot;\sigma^{*,n})||_{L^\infty(\mathbb{S})} = o(1)$. By definition of e^n,

$$||e^n(\cdot;\sigma^{*,n})||_{L^\infty(\mathbb{S})} \leq 2||\int_{\mathbb{R}} \nu F^n_\nu(d\nu|s)||_{L^\infty(\mathbb{S})}.$$

From Assumption F.1, the RHS is of order $o(1)$.

Regarding equation (28), since $U(\cdot,\sigma^*,\mathbf{B}^*(s;q^*);q^*)$ is uniformly bounded and continuous (see Lemma F.1), the desired result follows from Lemma F.5(2).

Regarding equation (29), note that

$$\begin{aligned}&\sup_B |\mathbb{M}_{\sigma^{*,n},\mathbf{B}^{*,n}}[U(\cdot,\sigma^{*,n},\mathbf{B}^{*,n}(s;q^{n,*});q^{n,*})-U(\cdot,\sigma^*,\mathbf{B}^*(s;q^*);q^*)](s)|\\ \leq & ||\mathbb{M}_{\sigma^{*,n},\mathbf{B}^{*,n}}|| \times \|U(\cdot,\sigma^{*,n},\mathbf{B}^{*,n}(s;q^{n,*});q^{n,*})-U(\cdot,\sigma^*,\mathbf{B}^*(s;q^*);q^*)](s)\|_{L^\infty(\mathbb{S})}.\end{aligned}$$

Since $||\mathbb{M}_{\sigma^{*,n},\mathbf{B}^{*,n}}|| \leq (1-\beta)$ (see above), it suffices to show that

$$\|U(\cdot,\sigma^{*,n},\mathbf{B}^{*,n}(s;q^{n,*});q^{n,*})-U(\cdot,\sigma^*,\mathbf{B}^*(s;q^*);q^*)](s)\|_{L^\infty(\mathbb{S})}=o(1).$$

By similar algebra to that in equation (21), the previous expression can be bounded above by

$$\sup_{y,B,B'}\left|\frac{du(y+B-\bar{q}^{*,n}(y,B')B')}{dc}\right|\{\gamma\left\|\mathbf{B}^*-\mathbf{B}^{n,*}\right\|_{L^\infty(\mathbb{S})}+\bar{B}\left\|\mathbf{q}^*-\mathbf{q}^{n,*}\right\|_{L^\infty(\mathbb{S})}\};$$

since $q^{n,*}\leq\gamma$ (see equation (18)) and $\bar{B}<\infty$ (because $\mathbb{B}$ is bounded). Under Assumption F.2 and hence, the previous expression is of order $O\left(\max\{\left\|\mathbf{q}^*-\mathbf{q}^{n,*}\right\|_{L^\infty(\mathbb{S})},\left\|\mathbf{B}^*-\mathbf{B}^{n,*}\right\|_{L^\infty(\mathbb{S})}\}\right)$. □

Let $\mathbb{C}(\mathbb{S})$ be the space of continuous and uniformly bounded functions that map $\mathbb{S}$ onto $\mathbb{R}$.

LEMMA F.5. *For any* $(\sigma,\mathbf{B})$*: (1)*

$$\left(\sum_{t=0}^{\infty}\beta^t\mathbb{L}^t_{\sigma,\mathbf{B}}\right)=\left(I-\beta\mathbb{L}_{\sigma,\mathbf{B}}\right)^{-1}$$

exists as an element of $\mathcal{L}(L^\infty(\mathbb{S}))$*; (2) For any* $g\in\mathbb{C}(\mathbb{S})$ *and* s, $(\sigma,\mathbf{B})\mapsto\left(I-\beta\mathbb{L}_{\sigma,\mathbf{B}}\right)^{-1}[g](s)$ *is continuous under the* $L^\infty(\{0,1\}\times\mathbb{S})\times L^\infty(\mathbb{S}\times\{0,1\})\setminus|\cdot|$ *norms.*

Proof of Lemma F.5. (1) We note that $||\mathbb{L}_{\sigma,\mathbf{B}}||\leq 1$; hence $||\sum_{t=0}^{T}\beta^t\mathbb{L}^t_{\sigma,\mathbf{B}}||\leq\sum_t^T\beta^t||\mathbb{L}^t_{\sigma,\mathbf{B}}||\leq(1-\beta)^{-1}$. It is easy to see that this implies that the partial sums, $(\sum_{t=0}^{T}\beta^t\mathbb{L}^t_{\sigma,\mathbf{B}})_T$, are a Cauchy sequence, and since $\mathcal{L}(L^\infty(\mathbb{S}))$ is complete, it converges (to S). Pre-multiplying by $\beta\mathbb{L}_{\sigma,\mathbf{B}}$, it follows that

$$\beta\mathbb{L}_{\sigma,\mathbf{B}}S=\sum_{t=1}^{\infty}\beta^t\mathbb{L}^t_{\sigma,\mathbf{B}}=S-I.$$

Hence $I=S(I-\beta\mathbb{L}_{\sigma,\mathbf{B}})$. Post-multiplying by $\beta\mathbb{L}_{\sigma,\mathbf{B}}$ an analogous result follows and thus the desired result is proven.

(2) We want to show that, for any $\epsilon>0$, there exists $\eta>0$ such that

$$\sup_B\left|\left(\sum_{t=0}^{\infty}\beta^t\mathbb{L}^t_{\tilde{\sigma},\tilde{\mathbf{B}}}-\sum_{t=0}^{\infty}\beta^t\mathbb{L}^t_{\sigma,\mathbf{B}}\right)[g](s)\right|<\epsilon, \tag{30}$$

for all $(\tilde{\sigma}, \tilde{\mathbf{B}})$ such that $||\tilde{\sigma} - \sigma||_{L^\infty(\{0,1\}\times\mathbb{S})} + ||\tilde{\mathbf{B}} - \mathbf{B}||_{L^\infty(\mathbb{S})} \leq \eta$.

The LHS of equation (30), is bounded above by

$$\sup_B \left| \left(\sum_{t=0}^{T} \beta^t \mathbb{L}^t_{\tilde{\sigma},\tilde{\mathbf{B}}} - \sum_{t=0}^{T} \beta^t \mathbb{L}^t_{\sigma,\mathbf{B}} \right) [g](s) \right| + \left\| \sum_{t=T+1}^{\infty} \beta^t \mathbb{L}^t_{\tilde{\sigma},\tilde{\mathbf{B}}} - \sum_{t=T+1}^{\infty} \beta^t \mathbb{L}^t_{\sigma,\mathbf{B}} \right\| ||g||_{L^\infty(\mathbb{S})}.$$

Since $||g||_{L^\infty(\mathbb{S})} < \infty$ and $||\mathbb{L}_{\sigma,\mathbf{B}}|| \leq 1$ for all $(\sigma, \mathbf{B})$, it follows there exists a $T(\epsilon)$ such that the second summand is less than 0.5ϵ. Since $T(\epsilon) < \infty$, to show that the first summand is less than 0.5ϵ, it suffices to show that there exists a $\eta > 0$, such that

$$\sup_B \left| \left(\mathbb{L}_{\tilde{\sigma},\tilde{\mathbf{B}}} - \mathbb{L}_{\sigma,\mathbf{B}} \right) [g](s) \right| < \epsilon, \tag{31}$$

for all $(\tilde{\sigma}, \tilde{\mathbf{B}})$ such that $||\tilde{\sigma} - \sigma||_{L^\infty(\{0,1\}\times\mathbb{S})} + ||\tilde{\mathbf{B}} - \mathbf{B}||_{L^\infty(\mathbb{S})} \leq \eta$. We note that the δ will depend on $T(\epsilon)$ too.

We can cast the RHS of equation (31) as

$$\sup_B \left| \int_{\mathbb{Y}} \left\{ \sum_{\delta\in\{0,1\}} \left(\tilde{\sigma}(\delta|s) g(\tilde{\mathbf{B}}(\cdot), y', \delta) - \sigma(\delta|s) g(\mathbf{B}(\cdot), y', \delta) \right) \right\} f(y'|y) dy' \right|.$$

Since $||\tilde{\sigma} - \sigma||_{L^\infty(\{0,1\}\times\mathbb{S})} < \delta$ and $g \in L^\infty(\mathbb{S})$ —by the Dominated convergence theorem—it suffices to show, for all (δ, y) and $\epsilon > 0$, there exists a $\eta > 0$ (it may depend on (δ, y)) such that

$$\sup_B \left| g(\tilde{\mathbf{B}}(\cdot), y, \delta) - g(\mathbf{B}(\cdot), y, \delta) \right| < \epsilon,$$

for all $(\tilde{\sigma}, \tilde{\mathbf{B}})$ such that $||\tilde{\sigma} - \sigma||_{L^\infty(\{0,1\}\times\mathbb{S})} + ||\tilde{\mathbf{B}} - \mathbf{B}||_{L^\infty(\mathbb{S})} \leq \eta$.

To show this, we note that $B' \mapsto g(B', y, \delta)$ is continuous. Hence, for every B, there exists an $\eta_B(\epsilon, y, \delta)$ such that

$$\left| g(\tilde{\mathbf{B}}(B, y), y, \delta) - g(\mathbf{B}(B, y), y, \delta) \right| < \epsilon$$

for all $|\tilde{\mathbf{B}}(B, y) - \tilde{\mathbf{B}}(B, y)| < \eta_B(\epsilon, y, \delta)$. By choosing $||\tilde{\mathbf{B}} - \mathbf{B}||_{L^\infty(\mathbb{S})} < \eta(\epsilon, y, \delta)$, the previous equation ensures that

$$\sup_B \left| g(\tilde{\mathbf{B}}(B, y), y, \delta) - g(\mathbf{B}(B, y), y, \delta) \right| < \epsilon.$$

Letting $\eta \equiv \eta(\epsilon, y, \delta)$ ensures the desired result. □