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The Distress Premium Puzzle

Ali K. Ozdagli

Abstract:
Fama and French (1992) suggest that the positive value premium results from risk of financial distress. However, recent empirical research has found that financially distressed firms have lower stock returns, using empirical estimates of default probabilities. This paper reconciles the positive value premium and the negative distress premium in a model that decouples actual and risk-neutral default probabilities. Moreover, in agreement with the data, firms with higher bond yields have higher stock returns in the model. The model also captures the fact that book-to-market value dominates financial leverage in explaining stock returns. Finally, the model predicts that firms with higher risk-neutral default probabilities should have higher stock returns, a hypothesis that can be tested using credit default swap premiums.

JEL Classifications: G12, G32, G33

Ali Ozdagli is an economist in the research department of the Federal Reserve Bank of Boston. His e-mail address is ali.ozdagli@bos.frb.org.

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1 Introduction

The conventional wisdom suggests that firms with high exposure to risk should have high expected returns and low market values, and be closer to default as a result of the latter. Consequently, firms' default probabilities should be positively correlated with market-based risk characteristics, such as dividend-price, earnings-price, and book-to-market ratios, and firms that are more likely to default should have higher expected equity returns. Indeed, Fama and French (1992) claim that size and value premiums result from distress risk.[1] However, recent empirical research, including Dichev (1998), Griffin and Lemmon (2002), and Campbell, Hilscher, and Sziglayi (henceforth, CHS) (2008), has reached the opposite conclusion that financially distressed firms have lower returns, using empirical estimates of default probabilities. Table 1 summarizes the evidence.

This paper argues that these patterns are not so puzzling once we realize that the default measures discussed above aim to capture the probability of observing a default under the real probability measure, and that this probability does not necessarily line up with the risk-neutral default probability that governs the market value of equity and the risk characteristics based thereon. We cannot back out risk-neutral default probabilities using default observations from the data even if we have the perfect model because we are trying to fit the econometric model to observed defaults. The discrepancy between real and risk-neutral probability measures can explain the patterns observed above.

A simple example clarifies this point. Let x be a stock's payoff tomorrow, that is, its future price plus dividends, and let m be the discount factor. Then, today's price is given by $p = E[mx]$ and, using the gross risk-free rate $R_f = 1/E[m]$, the expected excess return of stock is given by

$$\frac{E[R_e]}{R_f} = \frac{E[x]}{E[R_f mx]} = \frac{E[x]}{E^*[x]}. \tag{1}$$

[1] The positive relationship between stock returns and market-based risk charateristics is well documented. See Fama and French (1992) for the relationsship between B/M and returns. See Ball (1978) and Lettau and Wachter (2007) for the relationship between D/P, E/P, and returns.

Table 1: **Average stock returns in different portfolios.** The portfolio returns for earnings-price (E/P) and book-to-market (B/M) ratios are adapted from Table I of excess returns in Lettau and Wachter (2007). The returns for distress portfolios based on O-score are adapted from Table IV of Dichev (1998) and those based on CHS-score (CHS) are adapted from Table VI of Campbell, Hilscher, Szilagyi (2008). All returns are modified using the monthly T-bill and market return series in Ken French's website and then multiplied by 12 so that all returns are annualized actual returns rather than excess returns. Note two things: First, the distress premium implied by O-score and CHS-score are different because the default frequency in the data is low so that the empirical estimates of default probabilities can vary significantly across different methodologies. Second, all portfolios except CHS are constructed by sorting the firms into deciles, whereas the CHS portfolios include the following percentiles: 0-5, 5-10, 10-20, 20-40, 40-60, 60-80, 80-90, 90-95, 95-99, 99-100.

Portfolio	E/P	B/M	O-Score	CHS
1	9.21	10.17	14.16	13.30
2	9.52	11.05	15.12	11.48
3	11.47	11.48	15.72	10.97
4	11.54	11.01	16.32	10.93
5	11.50	12.50	15.36	10.58
6	13.68	12.83	14.88	9.77
7	14.44	12.77	15.00	5.49
8	15.68	14.58	15.48	2.03
9	16.18	14.48	13.44	3.20
10	17.45	15.05	7.20	-6.14

The denominator of this expression is the expected payoff under the risk-neutral probability measure, whereas the numerator is the expected payoff under the real measure.[2] If the risk-neutral and real probability distributions do not comove perfectly across firms or across time, then $E[x]$ and $E^*[x]$ will be only weakly correlated. In addition, because the market value of equity, $E[mx]$, is determined by the risk-neutral probability measure, market-based risk characteristics will be weakly correlated with real default probabilities determined by the real measure. Similarly, the expected payoff, $E[x]$, is determined by the real probability measure and will be weakly correlated with risk-neutral default probabilities.

Moreover, if we let DF denote the real default probability, we can take the logarithms and then take the derivative of both sides of equation (1) to get

$$\frac{d\ln E[R_e]}{dDF} = \frac{d\ln E[x]}{dDF} - \frac{d\ln E[mx]}{dDF}.$$

Intuitively, firms with lower real default probabilities are more likely to have higher expected payoffs under the real probability measure, $E[x]$, and hence the first term should be negative. However, the weak correlation between $E[mx]$ and the real default probabilities implies that the second term is only weakly negative. As a result, expected returns decrease with observed likelihood of default.

A similar argument can be made for risk-neutral default probabilities. If we let DF^* denote the risk-neutral default probability, we can take the logarithms and then the derivative

[2] Let $f(x,m)$ denote the joint distribution of x and m. Then we can write the real and risk-neutral probabilities as

$$\begin{aligned}\pi(x) &= \int_{m\in M} f(x,m)\,dm \\ \pi^*(x) &= \int_{m\in M} f(x,m)\frac{m}{E(m)}dm.\end{aligned}$$

Note that both of these quantities are probability measures, that is, they satisfy $\pi > 0$ and $\int \pi = 1$. Hence, the numerator is $E[x] = \int \pi(x)\,x dx$ and the denominator is $E^*[x] = \int \pi^*(x)\,x dx$.

of both sides of equation (1) to get

$$\frac{d\ln E[R_e]}{dDF^*} = \frac{d\ln E\,[x]}{dDF^*} - \frac{d\ln E\,[mx]}{dDF^*}.$$

Intuitively, firms with lower risk-neutral default probabilities are more likely to have higher expected payoffs under the risk-neutral probability measure, $E\,[mx]$, and hence the second term should be negative. However, the weak correlation between $E\,[x]$ and risk-neutral default probabilities implies that the first term is only weakly negative. As a result, expected returns increase with the risk-neutral likelihood of default.

The bottom line of this hypothesis is that both Fama and French and the studies that find a negative relationship between stock returns and likelihood of default are right. On the one hand, as empirical studies suggest, firms with a higher observed likelihood of default should have lower returns, given risk-neutral default probabilities. On the other hand, firms with a higher default probability under the risk-neutral measure should have higher market-based risk characteristics and higher returns, given observed default probabilities.

This is not the first paper that tries to explain the distress premium in a theoretical model. George and Hwang (2009) show in a static model that high distress cost leads to low leverage and default probability and higher returns for the total (unlevered) firm.[3] Garlappi, Shu, and Yan (2007) show that violation of the absolute priority rule in bankruptcies creates a hump-shaped relationship between expected equity returns and default probability. However, Bharath, Panchapagesan, and Werner (2007) state that there has been a secular decline in the frequency of absolute priority deviations (APDs) in favor of shareholders: APDs were commonplace, as high as 75 percent, before 1990; 22 percent in the 1991-2005 period, and as low as 9 percent for the period 2000-2005. This finding puts a question mark on the explanation of the distress premium via APDs, since the negative distress premium seems

[3]The proof of proposition 1 in George and Hwang's paper is presented for the expected returns on the total firm value, not for stock returns, although their main aim is to explain the negative distress premium in stock returns. It seems that their mechanism does not necessarily carry over to stock returns, that is, their mechanism does not solve the negative distress premium puzzle for stock returns.

to persist after 1990. Avramov, Cederburg, and Hore (2010) relate the negative distress premium to long-run risk, which seems quite intuitive, but they lose the negative relationship between price-earnings ratios and returns observed in the data: In their model, an increase in relative shares ($\bar{\theta}/\theta$ in their paper) increases the price-dividend ratios (equation 13 in their paper) and increases expected returns (Figure 2(a) in their paper), which implies a positive relationship between P/D and returns. This contradicts the data presented in Lettau and Wachter (2007).

The criticism to Avramov, Cederburg, and Hore (2010) can also be generalized to George and Hwang (2009), Garlappi, Shu, and Yan (2007) and Garlappi and Yan (2010). In all of these papers, the risk-neutral and real default probabilties are monotonically related. Therefore, the market-based risk characteristics that are governed by risk-neutral default probabilities are highly correlated with real default probabilities, and a negative distress premium should imply a negative value premium. This creates another challenge, which is the need to capture simultaneously the negative distress premium and the positive value premium.

To summarize, this paper tries to capture the following three regularities observed in the distress premium literature:

1- Firms with higher default likelihood have lower returns, as discussed in CHS (2009) and the references therein.

2- Firms with higher earnings-price ratios and higher book-to-market values have higher returns, as discussed in Fama and French (1992) and Lettau and Wachter (2007).

3- When the firms are ranked according to their bond yields, firms with higher bond yields have higher returns. Indeed, Anginer and Yildizhan (2010) find that firms with higher bond yields have higher stock returns.

Aside from matching these regularities, Section 5 shows that the model predicts that firms with higher default probability under a risk-neutral measure should have higher returns, a prediction that can be tested using the risk-neutral default probabilities implied by credit default swaps. Moreover, Section 6 shows that the model successfully captures the following patterns that involve book-to-market value, financial leverage, and stock returns:

1- Stock returns are positively related with market leverage (Bhandari (1988), Fama and French (1992), Gomes and Schmid (2010)) but are insensitive to book leverage (Gomes and Schmid (2010)).

2- Stock returns are less sensitive to market leverage than to book-to-market ratios.

3- Market leverage is only weakly linked to stock returns after controlling for book-to-market value. (Johnson (2003), Gomes and Schmid (2010))

4- Book leverage remains insensitive to stock returns after controlling for book-to-market value but becomes negatively related to stock returns after controlling for market leverage. (Fama and French (1992))

2 The Model

This section presents investors' preferences and the dynamic model of the firm, where the levels and riskiness of the cash flows, the latter measured by the exposure of cash-flow growth to systematic risk, are the only sources of heterogeneity.[4] Therefore, although a reduction in cash flow will increase the default probability, the risk-neutral and risk-free default probabilities will not comove perfectly across firms because of heterogeneity in the riskiness of their cash flows.

The investor's preferences for intertemporal substitution and risk are given by a constant risk-free interest rate, r, and price of risk, σ_S:

$$\frac{d\Lambda}{\Lambda} = -rdt - \sigma_S dw_A,$$

where Λ_{t+s}/Λ_t is the stochastic discount factor and dw_A is a Brownian increment that cap-

[4] The heterogeneity of cash-flow riskiness is a realistic assumption. Some companies, like fast food and dollar store chains, have less cyclical earnings than their more upscale counterparts. Good examples are McDonald's (MCD) and Family Dollar Stores (FDO) versus Ruby Tuesday (RT) and Kohl's (KSS).

tures macroeconomic shocks.

2.1 Equity Valuation

Firm i's cash flow, X_i, follows a geometric Brownian motion

$$\frac{dX_i}{X_i} = \mu_X dt + \sigma \left(\rho_i dw_A + \sqrt{1-\rho_i^2} dw_i \right),$$

where μ_X and σ are the growth rate and volatility of cash flow, assumed to be the same across firms for the sake of parsimony, and ρ_i captures the heterogeneity of cash flow's riskiness across firms.

Similar to its formulation in He and Xiong (2010), the debt takes the form of a coupon bond with a maturity date arriving at an exogenously given rate, λ, and the debt level is chosen optimally by the firm at date zero. Upon maturity of existing debt, the firm can choose to refinance by paying off the existing debt and issuing new debt or to go bankrupt, leaving the ownership of the firm to the lenders, who incur a bankruptcy cost proportional to the after-tax value of the firm, given by $(1-\eta)$. Similar to Fisher, Heinkel, and Zechner (1989), I assume that the cost of issuing new debt is proportional to the size of the new issue, with b the proportionality factor of the cost of issuing new debt, and that the debt is issued at par value.

The assumption regarding debt maturity ensures that relatively few firms are close to the endogenous default boundary, so that the equity betas of the most distressed firms do not explode, as discussed in Garlappi and Yan (2010). Having a fixed maturity date would serve the same purpose and would not change the results qualitatively. However, a fixed maturity date would make solution of the model harder because time would enter the model as a state variable. The debt structure in this paper generates the time homogeneity of the problem and allows for closed-form solutions. An alternative interpretation of this assumption is that the firm issues short-term debt that gets rolled over at the same coupon rate in each time period $(t, t+dt)$, with probability $(1-\lambda dt)$. This interpretation is similar to the one Leland (1994a, p. 1215) proposes for infinite maturity debt, which is a special case of this model with $\lambda = 0$. Other time-homogeneous settings are presented in Leland (1994b) and Leland and Toft (1996). However, in both models debt is issued continuously, which contradicts Welch's (2004) findings that firms change their debt levels infrequently in response to changes in their stock prices.

If we let instantenous coupon payment be c, the corporate tax rate be τ, and the market value of debt be $B(X,c)$, we can write the Hamilton-Jacobi-Bellman (HJB, henceforth) equation for the market value of equity, $J(X,c)$, as

$$\begin{aligned} rJ(X,c) &= (1-\tau)(X-c) + \mu X J_X(X,c) + \frac{1}{2}\sigma^2 X^2 J_{XX}(X,c) \\ &\quad + \lambda \left(\max \left\{ 0, \max_{c'} J(X,c') + (1-b) B(X,c') - B(X_0,c) \right\} - J(X,c) \right), \end{aligned}$$

where $\mu = \mu_X - \rho\sigma\sigma_S$ is the risk-adjusted drift of the cash-flow process and X_0 is the value of cash flow at the time of the last debt issue.[5] Since debt is issued at par by assumption, $B(X_0,c)$ is equal to the par value of debt. The firm-specific indices are dropped for parsimony from here on.

The firm chooses its default boundary, X_B, optimally so that $J(X,c)$ satisfies the value matching and smooth pasting conditions

$$J(X_B,c) = J_X(X_B,c) = J_c(X_B,c) = 0.$$

[5] I omit the personal income tax and assume full loss offset via taxes, as is customary in the literature. For recent examples, see Maio (2005) and Chen (2010). These assumptions can be relaxed without changing any result qualitatively.

2.2 Debt Valuation

Noting that the after tax value of debt upon bankruptcy is $\frac{\eta(1-\tau)}{r-\mu}X$, one can write the HJB equation for the market value of debt as

$$\begin{aligned} rB(X,c) &= c+\mu XB(X,c)+\frac{1}{2}\sigma^2X^2B_{XX}(X,c) \\ &\quad +\lambda\left((1-\mathbb{I}_B)B(X_0,c)+\mathbb{I}_B\frac{\eta(1-\tau)}{r-\mu}X-B(X,c)\right) \end{aligned}$$

with the boundary condition

$$B(X_B,c)=\frac{\eta(1-\tau)}{r-\mu}X_B,$$

where $\mathbb{I}_B$ is the indicator function that is equal to 1 if the firm prefers bankruptcy at debt maturity and zero if the firm chooses to refinance, that is,

$$\mathbb{I}_B=\begin{cases} 1 \text{ if } \max_{c'} J(X,c')+(1-b)B(X,c')-B(X_0,c)\leq 0 \\ 0 \text{ otherwise} \end{cases}.$$

3 Solution

3.1 Homogeneity of Market Values

I will solve the joint problem of the firm and debtholders, using the guess and verify technique. Since the payoffs and boundary conditions are homogeneous in X and c, I start with the guess that both $J(X,c)$ and $B(X,c)$ are linearly homogeneous in X and c.[6] Define $y\equiv c/X$, $E(y)\equiv J(X,c)/X$, and $D(y)\equiv B(X,c)/X$.[7] Then the HJB for market value of equity

[6] It is possible to use a more direct solution since the HJB equations for equity and debt can be treated as ordinary differential equations whose solution has constants of integration that depend on c. However, the method presented here illuminates the intuition regarding the optimal behavior of the firm addressed in the next section.

[7] The variable y also has an economic meaning: it is the interest coverage ratio. Moreover, $E(y)$ is the P/E ratio.

becomes

$$\begin{aligned}
(r-\mu) E(y) &= (1-\tau)(1-y) - \mu y E'(y) + \frac{1}{2}\sigma^2 y^2 E''(y) \\
&\quad + \lambda \left(\max\left\{ 0, \max_{y'} E(y') + (1-b) D(y') - \frac{B(X_0, c)}{X} \right\} - E(y) \right)
\end{aligned}$$

with the boundary conditions

$$E(y_B) = E'(y_B) = 0,$$

where $y_B \equiv c/X_B$.

Similarly, the HJB for debt becomes

$$\begin{aligned}
(r-\mu) D(y) &= y - \mu y D'(y) + \frac{1}{2}\sigma^2 y^2 D''(y) \\
&\quad + \lambda \left((1-\mathbb{I}_B) \frac{B(X_0, c)}{X} + \mathbb{I}_B \frac{\eta(1-\tau)}{r-\mu} - D(y) \right)
\end{aligned}$$

with boundary condition

$$D(y_B) = \frac{\eta(1-\tau)}{r-\mu}$$

and the indicator function

$$\mathbb{I}_B = \begin{cases} 1 \text{ if } \max_{y'} E(y') + (1-b) D(y') - \frac{B(X_0,c)}{X} \leq 0 \\ 0 \text{ otherwise} \end{cases}.$$

Finally, we can verify the guess by showing that both of the HJB equations can be represented in terms of y, y_B, $E(y)$, and $D(y)$. Since the only term that does not depend exclusively on y in these equations is $\frac{B(X_0,c)}{X}$, we will focus on this term. Before doing that, let us define y_0 as

$$y_0 \equiv \arg\max_{y'} E(y') + (1-b) D(y')$$

and note that y_0 is a constant number because neither $E(y)$ nor $D(y)$ depends explicitely on time. This implies that the firm chooses the same value of $y = y_0$ whenever it issues new debt, including the time of its inception. It follows that $c/X_0 = y_0$, since X_0 is the value of the cash flow at the time of the last debt issue and c is the current value of the coupon payment determined at the time of last issue. Using this result and our guess of homogeneity of the debt function $B(X, c)$, we get

$$\frac{B(X_0, c)}{X} = \frac{X_0 D(y_0)}{X} = \frac{X_0/c}{X/c} D(y_0) = \frac{y}{y_0} D(y_0),$$

and hence the term $\frac{B(X_0,c)}{X}$ can be expressed as a linear function of y. Substituting $\frac{B(X_0,c)}{X}$ with $\frac{y}{y_0} D(y_0)$ verifies the intial guess of homogeneity, as both HJB equations can be represented in terms of functions of y. In particular, the indicator function becomes

$$\mathbb{I}_B = \begin{cases} 1 \text{ if } E(y_0) + (1-b) D(y_0) - \frac{y}{y_0} D(y_0) \leq 0 \\ 0 \text{ otherwise} \end{cases},$$

which we will use in the next section.

3.2 Optimal Policy of the Firm

In this section we characterize basic properties of the optimal policy of the firm and leave the complete characterization to the appendix. We have seen that the firm chooses bankruptcy whenever refinancing provides a non-positive value to the shareholders, that is,

$$S(y) \equiv E(y_0) + (1-b) D(y_0) - \frac{y}{y_0} D(y_0) \leq 0.$$

The following proposition characterizes the optimal behavior of the firm at the time of debt maturity.

Proposition 1 *There is a threshold level of y, denoted as $\bar{y}$, above (below) which the firm chooses bankruptcy (refinancing) at the time of debt maturity.*

Proof. Note that $y > 0$ and $\lim_{y \to 0^+} S(y) = E(y_0) + (1-b) D(y_0) > 0$, because if $E(y_0) + (1-b) D(y_0) \leq 0$ the firm would choose not to enter the market at its inception. Moreover,

$S'(y) < 0$ and $\lim_{y->\infty} S(y) = -\infty$. Therefore, by the intermediate value theorem, there exists a unique $\bar{y} > 0$ that satisfies $S(\bar{y}) = 0$ and $S(y) \lesseqgtr 0$ if $y \gtreqless \bar{y}$. Since $S(y) \leq 0$ $(S(y) > 0)$ implies bankruptcy (refinancing) this completes the proof. ■

This proposition tells us that the firm chooses bankruptcy at debt maturity if its cash flow is low so that the shareholders rather pass on the ownership of the firm to the lenders, and that the firm chooses to refinance its debt if its cash flow is high enough so that the net value of restructuring is positive to the shareholders. If the firm prefers to refinance, it chooses its debt level so as to maximize its shareholder value, that is, the new coupon payment is chosen so that it is equal to $y_0 X$. Therefore, default occurs either because the firm is insufficiently productive at the time of debt maturity or because the firm's cash flow hits the endogenous default boundary before debt maturity.

The following proposition and its corollary show the relative positioning of y_0, $\bar{y}$, and y_B when the cost of debt issuance is small, thereby refining the properties of optimal policy.

Proposition 2 *In the absence of debt issuance costs, that is $b = 0$, $\bar{y}$ satisfies $y_0 < \bar{y} < y_B$.*

Proof. See Appendix ■

The following corollary follows from the fact that the value functions and boundary conditions are continuous and differentiable in b.

Corollary 3 *For sufficiently small cost of issuing debt, $y_0 < \bar{y} < y_B$.*

According to my analysis, the choice of debt issuance cost, b, in my calibration is small enough so that $y_0 < \bar{y} < y_B$. Therefore, the rest of the analysis in the paper is based on the case $y_0 < \bar{y} < y_B$, although the intuition derived from the analysis would be similar under different scenarios.

4 Stock Returns, Default Probabilities, and the Long-Run Distribution of Firms

The instantanous expected excess stock returns are given by the sum of dividends and capital appreciation divided by the current value of the firm under the real measure minus the risk-free rate,

$$\begin{aligned}
\mathbb{E}_t\left(dR_i^e\right) - rdt &= \mathbb{E}_t\left(\frac{(1-\tau)\left(X_i - c\right)dt + dJ_i}{J_i}\right) - rdt \\
&= \rho_i \sigma \sigma_S \frac{J_{iX} X_i}{J_i} dt = \rho_i \sigma \sigma_S \left(1 - \frac{E_i'\left(y_i\right) y_i}{E_i\left(y_i\right)}\right) dt.
\end{aligned}$$

The first equality in the second line comes from the HJB equation for the market value of equity and the relationship between the real and the risk-netural drift of cash flow. The second equality comes from the homogeneity property of the market value of equity.

As discussed in the previous section, the firm defaults when its cash flow hits the endogenous default boundary, $y = y_B$, or when $y > \bar{y}$ at the time of debt maturity. The appendix shows that the moment generating function for the distribution of time to default is given by

$$M\left(s; y\right) = \begin{cases} A_m \left(\frac{\lambda}{-s}\left(\frac{y_0}{y_B}\right)^{\theta_2} + \left(\frac{y}{y_B}\right)^{\theta_2}\right) & \text{if } y \leq \bar{y} \\ \frac{\lambda}{-s+\lambda} + \frac{-s}{-s+\lambda}\left(\frac{y}{y_B}\right)^{\theta_2} + B_m\left(\left(\frac{y}{y_B}\right)^{\theta_1} - \left(\frac{y}{y_B}\right)^{\theta_2}\right) & \text{otherwise} \end{cases},$$

where $\theta_2 > 0 > \theta_1$ are the roots of

$$\frac{1}{2}\sigma^2\theta^2 + m\theta - (s + \lambda) = 0$$

and A_m and B_m satisfy

$$\begin{aligned}\left(\frac{\lambda}{-s}\left(\frac{y_0}{y_B}\right)^{\theta_2}+\left(\frac{\bar{y}}{y_B}\right)^{\theta_2}\right)A_m &= \frac{\lambda}{-s+\lambda}+\frac{-s}{-s+\lambda}\left(\frac{\bar{y}}{y_B}\right)^{\theta_2}+B_m\left(\left(\frac{\bar{y}}{y_B}\right)^{\theta_1}-\left(\frac{\bar{y}}{y_B}\right)^{\theta_2}\right)\\ \theta_2\left(\frac{\bar{y}}{y_B}\right)^{\theta_2}A_m &= \frac{-s}{-s+\lambda}\theta_2\left(\frac{\bar{y}}{y_B}\right)^{\theta_2}+B_m\left(\theta_1\left(\frac{\bar{y}}{y_B}\right)^{\theta_1}-\theta_2\left(\frac{\bar{y}}{y_B}\right)^{\theta_2}\right).\end{aligned}$$

Since y and ρ are the only source of heterogeneity across firms, we can index the firms by (y,ρ). The appendix shows that the long-run distribution of firms for a given cash-flow riskiness ρ is

$$\varphi\left(y|\rho\right)=\begin{cases}\bar{A}/y\left(\frac{y}{y_B}\right)^{\alpha_2} & \text{if } y\leq y_0\\ \bar{B}/y\left(\left(\frac{y}{y_B}\right)^{\alpha_1}-\left(\frac{y}{y_B}\right)^{\alpha_2}\right) & \text{if } y_0<y\leq y_B\end{cases},$$

where $\alpha_1<0<\alpha_2$ are the roots of

$$\frac{1}{2}\sigma^2\alpha^2+\left(\mu_X-\frac{1}{2}\sigma^2\right)\alpha-\lambda=0$$

and $\bar{A}$ and $\bar{B}$ satisfy

$$\begin{aligned}\bar{A}\alpha_2\left(\frac{y_0}{y_B}\right)^{\alpha_2}-\left[\alpha_1\left(\left(\frac{y_0}{y_B}\right)^{\alpha_1}-1\right)-\alpha_2\left(\left(\frac{y_0}{y_B}\right)^{\alpha_2}-1\right)\right]\bar{B} &= \frac{2\lambda}{\sigma^2}\\ \frac{\left(\frac{y_0}{y_B}\right)^{\alpha_2}}{\alpha_2}\bar{A}-\left(\frac{\left(\frac{y_0}{y_B}\right)^{\alpha_1}-1}{\alpha_1}-\frac{\left(\frac{y_0}{y_B}\right)^{\alpha_2}-1}{\alpha_2}\right)\bar{B} &= 1.\end{aligned}$$

Therefore, the long-run distribution of the firms is given by

$$f\left(y,\rho\right)=\varphi\left(y|\rho\right)h\left(\rho\right),$$

where $h\left(\rho\right)$ is the distribution of ρ_i's assumed to be uniform between ρ_L and ρ_H, and $\varphi\left(y|\rho\right)$ is the distribution of firms whose cash-flow riskiness is measured by ρ.

5 Calibration and Stock Returns in the Long Run

I set $\mu_X = 0.02$ and $\sigma = 0.35$.[8] The tax rate is taken to be 35 percent from Taylor (2003) and Miao (2005). The risk-free rate is taken to be 2 percent, using the time series average of Fama's monthly T-bill returns in the CRSP database from 1963 to 2008. Moreover, I set $\sigma_S = 0.4$ in order to match the average monthly Sharpe ratio of the excess market return from 1963 to 2008. The cost of debt issuance is chosen to be the same value as in Fischer, Heinkel, and Zechner (1989), that is, $b = 0.01$. Following Huang and Huang (2003), I choose the bankruptcy cost to be half of the firm's value, that is, $\eta = 0.5$. The support of the distribution for the riskiness of cash flows is chosen in order to match the average P/E ratio of 15 and an equity premium of 6.5 percent. Accordingly, $\rho_L = 0.2$ and $\rho_H = 0.6$. I choose $\lambda = 1/3$, which implies an average debt maturity of three years, although I have tried various values between $\lambda = 1/4$ and $\lambda = 1/2$, and find that the results presented in this section are not affected qualitatively. Using the calibrated parameters, I calculate the long-run distribution of firms as discussed in the previous section and draw 40,000 points from this distribution. Then, I use these draws to calculate earnings-price ratios, distress measures, bond yields, and expected stock returns for different firms and to form portfolios. The results are presented in the top panel of Table 2.[9]

Distress: The moment-generating function for the time to default provides us several measures of distress. One way is to use a saddlepoint approximation in order to calculate the probability of default within one year, because the O-score and the CHS-score are based on estimates of the default probability within one year. However, the saddlepoint approxima-

[8] Miao (2005) and Cooper (2006) use 0.025 and 0.035 for μ_X, and 0.25 for σ, following the mean and standard deviation of earnings growth of S&P 500 firms. However, these numbers hold for total earnings of S&P 500 firms. Moreover, the S&P 500 firms are the ones with the largest capitalization traded in NYSE or NASDAQ, which implies that they had a particularly successful history. Therefore, calibrating these numbers from the S&P 500 universe would overestimate the growth rate and underestimate the volatility. My choice of parameter values aims to reduce this bias. The main results are unaffected qualitatively by this choice.

[9] I also repeat the analysis using simulation, rather than draws from the long-run distribution. Simulation results presented in the appendix, Table 5, are quantitatively similar to the ones discussed in the rest of the paper.

tions are potentially subject to significant numerical errors because the default probabilities within one year are very low.[10] Therefore, I use the moment generating function to calculate the exact value of the expected time to default as a proxy for financial distress. The first row of Table 2 provides the equally weighted average of stock returns when portfolios are formed according to decreasing expected time to default as a proxy under the real measure for increasing financial distress. We see that firms with greater financial distress earn lower stock returns in the model, implying that the model successfully captures the distress premium puzzle.

The negative distress premium results from the way the firms choose their capital structure. Debt is determined by the tradeoff of the tax advantage of debt versus bankruptcy costs under the risk-neutral measure. The tax advantage of debt results in higher leverage, whereas bankruptcy costs result in lower leverage. Firms with riskier cash flows have lower cash-flow growth and a higher default probability for a given level of debt under the risk-neutral measure. This increases expected bankruptcy costs for a given debt level and hence these firms choose lower debt. As a result, they have a lower probability of default under the real measure because their distance to default is greater. Therefore, when we rank the firms according to real default probabilities, the firms with higher rank are those with lower cash-flow risk and hence lower expected equity returns. This leads to a negative distress premium.

Risk-neutral Distress: I repeat the exercise above using the expected time to default under the risk-neutral measure. The second row of Table 2 provides the equally weighted average of stock returns when portfolios are formed according to decreasing expected time to default as a proxy under the risk-neutral measure. We see that the firms with greater financial distress under the risk-neutral measure earn higher stock returns in the model.

[10] See Campbell, Hilscher, and Sziglayi (2009) for evidence that the actual default probabilities are low. Because the default probabilities are so low, any other approximation method, such as simulated probabilities, will also be subject to significant errors. Nevertheless, the saddlepoint approximations with normal and inverse-Gaussian bases provide qualitatively similar results when I use them to approximate the probability of default within five years. The saddlepoint approximation is discussed in detail in the appendix.

Table 2: **Equally weighted annualized average of stock returns.** 40000 observations are drawn from the joint distribution of (ρ, y). Portfolios are ranked according to increasing price-earnings ratios, decreasing expected time to default and increasing bond yields as a proxy for financial distress under real and risk-neutral measures, increasing book-to-market values, market leverage and book leverage.

Portfolio Returns with Different Rankings					
A. Cash-Flow Model					
Portfolio	1	2	3	4	5
Distress	13.49	12.41	11.2	9.96	9.71
Risk-Neutral Distress	8.58	9.95	11.23	12.48	14.51
Earnings/Price	8.58	9.95	11.23	12.49	14.5
Bond Yield	10.31	10.67	11.19	11.45	13.14
B. Investment Model					
Portfolio	1	2	3	4	5
Book-to-Market	8.59	9.95	11.24	12.49	14.47
Market Leverage	10.98	11.2	10.87	11.	12.71
Book Leverage	12.04	11.78	11.05	10.45	11.42

Intuitively, firms with a higher risk-neutral default probability are those that have higher coupon payments relative to their cash flow, given cash-flow risk, or those that have higher cash-flow risk given level of cash flow and coupon payments. Both of these channels increase the riskiness of the firm's equity: The first one levers up the net income of the firm, whereas the second one increases the exposure of the firm to systematic risk. This prediction of the model can be tested using the implied risk-neutral default probabilities from credit default swap (CDS) data. Given that the CDS intruments are relatively new and currently do not cover the whole Compustat/CRSP universe, testing this hypothesis in the near future will be problematic. A preliminary analysis using bond yields as a proxy for risk-neutral default probability has been performed by Anginer and Yildizhan (2010), discussed at the end of this section.

Earnings-price ratio: I rank the firms according to their earnings-price ratio and form five portfolios. The third row of Table 2 provides the equally weighted average of expected stock returns in each of these portfolios. We see, in accordance with the evidence in Lettau and Wachter (2007), that the firms with a higher earnings-price ratio earn higher stock returns in the model.

Intuitively, a firm has a high earnings-price ratio either because its cash-flow risk is high or because it is close to default under the risk-neutral measure so that its market value is low. Both of these effects make equity riskier and hence increase expected stock returns.

Bond yields: Finally, I rank the firms according to their bond yields. The reason for this exercise is that Anginer and Yildizhan (2010) use bond yields as a proxy for financial distress under the risk-neutral measure and find that when the firms are ranked according to their bond yields, the firms with higher bond yields have higher stock returns, contradicting the evidence in the rest of the literature that estimates default probabilities under the real measure.[11] Since the bond yield is the internal rate of return of the bond under the counterfactual assumption that the firm does not go bankrupt, we have[12]

$$yield = \frac{c + \lambda B\left(X_0, c\right)}{B\left(X, c\right)} - \lambda = \frac{y}{y_0} \frac{y_0 + \lambda D\left(y_0\right)}{D\left(y\right)} - \lambda.$$

Note that at the date of bond issue, that is, when $X = X_0$, the yield is equal to $c/B\left(X_0, c\right)$, which is familiar to financial economists, since the yield of a bond issued at par is equal to the coupon yield at the time of issue.

The fourth row of Table 2 provides the equally weighted average of stock returns when portfolios are formed according to bond yields. We see that, in accordance with Anginer and Yildizhan (2010), the firms with greater bond yields earn higher stock returns in the model.

The firms with higher bond yields are those that have higher coupon yields and a higher risk-neutral default probability, because they have to compensate the lender more for each dollar they borrow. Moreover, a high risk-neutral default probability implies that these firms have higher market leverage, because their market value of equity is low relative to the

[11] See Campbell, Hilscher, and Szilagyi (2010) and the references therein.

[12] Let $\tilde{B}$ be the discounted value of the payoffs from holding the bond, assuming counterfactually that the bond does not default. Then, since the bond is issued at par, we can write

$$yield * \tilde{B} = c + \lambda \left(B\left(X_0, c\right) - \tilde{B}\right).$$

Solving this for $\tilde{B}$ and setting $\tilde{B} = B\left(X, c\right)$ gives the result above.

market value of their debt. Therefore, these firms' equity is riskier. This mechanism leads to a positive relationship between bond yields and equity returns.

Note that the stock return difference across bond yield portfolios is somewhat lower than the stock return difference across risk-neutral distress and book-to-market portfolios in the model, suggesting that we might need to come up with a clearer measure of risk-neutral distress than raw bond yields, such as the risk-neutral default probabilities implied by credit default swaps. To see, in the context of this model, why bond yields are not a pure measure of risk-neutral distress, note that we can write the bond yields as

$$yield = \frac{c}{B\left(X,c\right)} + \lambda\left(\frac{B\left(X_0,c\right)}{B\left(X,c\right)} - 1\right),$$

where the first term is the coupon yield and the second term captures capital losses (gains) by the bondholders as the cash flow changes. The coupon yields are closely related to the risk-neutral default probabilities, because higher cash-flow risk implies lower bond value and higher risk-neutral default probability. However, the cash-flow risk affects the par value, $B\left(X_0,c\right)$, and the market value, $B\left(X,c\right)$, of the bond the same way, limiting the effect of risk-neutral distress on the capital gain term. Moreover, a higher level of current cash flow, X, implies a higher market value of the bond, without affecting the par value, and firms with a higher value of cash flow are less likely to default under the real measure, given the cash-flow risk. Therefore, the second term is more closely related to the real default probability than to the risk-neutral default probability.

6 Book-to-Market Value, Financial Leverage, and Stock Returns

This section discusses the relationship between book-to-market value, financial leverage, and stock returns, and argues that the model can successfully generate the patterns involving these quantities. So far, I have focused only on the ability of the model to explain the negative distress premium and positive value premium simultaneously, in a cash-flow model. In order to be able to talk about book-to-market value and financial leverage, I need to model the amount of capital a firm chooses. The next subsection serves this aim, and the second subsection discusses the stock return patterns.

6.1 Extension with Investment and Value Premium

Although so far I have modeled the cash flow of the firm, modeling investment is a straightforward exercise, using arguments similar to those in Miao (2005). If we let δ be the depreciation of capital, which is tax-deductible, and r be the rental cost of capital, k, we can write the after-tax profit function of the firm as

$$\pi(k, z, c) = (1 - \tau)\left(z^{\alpha} k^{1-\alpha} - \delta k - c\right) - rk,$$

where $z^{\alpha} k^{1-\alpha}$ is the production function and z is the productivity of the firm, which follows geometric Brownian motion

$$\frac{dz}{z} = \mu_z dt + \sigma_z \left(\rho_i dw_A + \sqrt{1 - \rho_i^2} dw_i\right).$$

Then, similar to the treatment in Miao (2005), profit maximization implies the neoclassical investment rule that the marginal after-tax product of capital is equal to the user cost of capital,

$$(1 - \alpha) z^{\alpha} k^{-\alpha} = \frac{r}{1 - \tau} + \delta$$

or equivalently

$$k = \left(\frac{1 - \alpha}{r/(1 - \tau) + \delta}\right)^{1/\alpha} z.$$

Plugging this back into the profit function $\pi(k, z)$ gives the optimized profit function $\bar{\pi}(X, c) = (1-\tau)(X-c)$, where

$$X = \left[\left(\frac{1-\alpha}{r/(1-\tau)+\delta}\right)^{(1-\alpha)/\alpha} - \left(\delta + \frac{r}{1-\tau}\right)\left(\frac{1-\alpha}{r/(1-\tau)+\delta}\right)^{1/\alpha}\right] z,$$

which follows the geometric Brownian motion

$$\frac{dX}{X} = \mu_X dt + \sigma\left(\rho_i dw_A + \sqrt{1-\rho_i^2} dw_i\right),$$

where $\mu_X = \mu_z$ and $\sigma = \sigma_z$. Note that the after-tax profit function reduces to the original profit function when there is no investment and the cash-flow process is the same as before. Therefore, all the claims regarding the returns, financial distress, price-earnings ratios, and bond yields can be carried over to this model with investment.

The main advantage of this extension is that now we can calculate meaningful values for book-to-market ratios. This allows me to check whether the model can successfully generate the value premium as in Fama and French (1992) and to compare the power of book-to-market value in explaining stock returns with that of financial leverage. I focus on the value premium here and leave the comparison of book-to-market value to financial leverage to the next subsection.

Note that the book value of total assets is given by k, and hence $k - B(X_0, c)$ gives the book value of equity, which is measured as the book value of total assets minus the book value of debt, whereas $J(X, c)$ gives the market value of equity. Therefore, if we define

$$\kappa = \frac{1-\alpha}{\alpha\left(\frac{r}{1-\tau}+\delta\right)},$$

we can write book-to-market value as

$$\begin{aligned}\frac{BE}{ME} &= \frac{k - B(X_0, c)}{J(X, c)} = \frac{\kappa X - B(X_0, c)}{J(X, c)} \\ &= \frac{\kappa - y/y_0 D(y_0)}{E(y)}.\end{aligned}$$

Following Miao (2005) I set $\delta = 0.1$. Moreover, I set $\alpha = 0.05$.[13]

Using the last formula I calculate the book-to-market values and then sort the firms according to their book-to-market values. The first row in panel B of Table 2 provides the equally weighted average of stock returns in different book-to-market portfolios. We see that, in accordance with Fama and French (1992), the firms with greater book-to-market values earn higher stock returns in the model, that is, the model successfully captures the value premium.

The intuition for the book-to-market effect is similar to the intuition for the earnings-price ratios. Firms with high book-to-market ratios are those with high cash-flow risk or are more likely to default under the risk-neutral measure. The first effect increases the overall business risk of these firms, whereas the second effect implies that the market value of their equity is low relative to the market value of their debt, leading to high market leverage. Both of these effects make the equity of the firms with high book-to-market values riskier, so these firms have higher expected stock returns. This mechanism creates the value premium.

6.2 Leverage and Stock Return Patterns

Table 3 shows six regressions of stock returns on book-to-market values and different measures of financial leverage. The following summarizes the regression results and cites examples from previous literature that have similar findings:

[13] My choice of α can be justified using a decreasing returns to scale Cobb-Douglas production function with capital and labor inputs, where labor is optimized out. Miao (2005) sets $\alpha = 0.4$. However, this choice generates a significant number of negative book-to-market values in my model, which contradicts the data. The choice of α does not change the results qualitatively as long as we make sure that the book-to-market values are positive.

Table 3: **Fama-MacBeth regressions of stock returns on various variables.** Book-to-market value (BE/ME), market leverage (ML), and book leverage (BL) at the beginning of the portfolio formation period. The coefficients are the time series average of regression coefficients for July 1963 to June 2008, and the t-statistics (in parentheses) are the average regression coefficient divided by its time series standard error.

log BE/ME	log ML	log BL
0.42 (6.38)		
	0.18 (2.78)	
		-0.04 (0.57)
0.45 (8.05)	-0.06 (1.11)	
0.42 (6.54)		-0.09 (1.41)
	0.62 (6.12)	-0.78 (7.80)

1- Stock returns are positively related with market leverage (Bhandari (1988), Fama and French (1992), Gomes and Schmid (2010)), but are insensitive to book leverage (Gomes and Schmid (2010)).

2- Stock returns are less sensitive to market leverage than to book-to-market values.

3- Market leverage is only weakly linked to stock returns after controlling for book-to-market value (Johnson (2003), Gomes and Schmid (2010)).

4- Book leverage remains insensitive to stock returns after controlling for book-to-market value, but becomes negatively related to stock returns after controlling for market leverage (Fama and French (1992)).

Table 4 shows the model-generated regression results using simulations. The model seems to do a good job in capturing the regularities above.

How does the model generate the result that book-to-market values are a much stronger predictor of stock returns than financial leverage? In a model without heterogeneity in cash-flow risk, book-to-market value and market leverage are strongly correlated with each other, since firms with higher book-to-market value also have a higher real and risk-neutral

Table 4: **Fama-MacBeth Regressions with simulated data.** Book-to-market value (BE/ME), market leverage (ML), and book leverage (BL) at the beginning of portfolio formation period. The coefficients are the time series average of regression coefficients, and the t-statistics (in parentheses) are the average regression coefficient divided by its time series standard error. A total of 1200 firms are simulated over 1200 months 100 times, and the first 600 months are dropped to allow the simulations converge to a steady state. The reported statistics are averages across simulations.

log BE/ME	log ML	log BL
0.58 (11.38)		
	0.13 (6.98)	
		0.02 (0.15)
0.57 (11.45)	0.02 (2.04)	
0.58 (11.07)		0.04 (1.49)
	0.62 (19.27)	-0.43 (14.50)

default probability and default probabilities are strongly correlated with financial leverage. Therefore, book-to-market value has hardly any explanatory power above and beyond that of market leverage. However, when there is heterogeneity in cash-flow risk across firms, the cash-flow risk affects book-to-market values and market leverage in an opposite way. To see this within the context of the model, note that book-to-market value and market leverage are given by

$$\begin{aligned}
\frac{BE}{ME} &= \frac{k - B\left(X_0, c\right)}{J\left(X, c\right)} = \frac{\kappa - y/y_0 D\left(y_0\right)}{E\left(y\right)} \\
ML &= \frac{B\left(X_0, c\right)}{B(X_0, c) + J\left(X, c\right)} = \frac{y/y_0 D\left(y_0\right)}{y/y_0 D\left(y_0\right) + E\left(y\right)}.
\end{aligned}$$

Given the level of cash flow and coupon payments, higher cash-flow risk reduces the market value of equity. This affects book-to-market values and market leverage in the same direction, since equity is in the denominator of both quantities. However, higher cash-flow risk also decreases the value of the firm's debt by increasing risk-neutral default probabilities. This depresses market leverage but increases book-to-market values, as we can see from the equations above. Therefore, book-to-market values are more sensitive to a change in cash-flow risk than is market leverage. Since cash-flow risk is positively related to expected returns, book-to-market values are more strongly correlated with returns than is market leverage and hence book-to-market subsumes the effect of financial leverage.

This intuition also explains why we have a significantly negative sign on book leverage, after controlling for market leverage. Note that we can write book-to-market value as a combination of book leverage and market leverage, that is

$$\frac{BE}{ME} = \frac{ML}{1 - ML} / \frac{BL}{1 - BL}.$$

Because book-to-market values subsume the relationship of financial leverage with stock returns, a regression of returns on market leverage and book leverage should imply a significantly positive sign for market leverage and a significantly negative sign for book leverage.

In a recent paper, Gomes and Schmid (2010) argue that the relationship of growth options and stock returns can explain the weak link observed between market leverage and stock returns. In their model, the growth options provide an additional source of risk for small young firms, and these firms have lower market leverage than large mature firms. Therefore, although the Modigliani and Miller paradigm suggests that firms with lower market leverage should have smaller equity risk, growth options affect this relationship in the opposite direction, resulting in an ambiguous relationship between market leverage and stock returns. My explanation is based on the heterogeneity of cash-flow riskiness, and my approach has important differences: First, in Gomes and Schmid, high book-to-market firms also have high default probabilities and they link the relationship between book-to-market values and stock returns to financial distress.[14] This implies that their model suggests a positive distress premium, in contrast with the evidence in CHS (2009) and the references therein. In com-

[14] Gomes and Schmid (2010, p. 490): "First, our dynamic model of leverage and returns offers theoretical support to the common intuition that book-to-market value is related to a financial distress factor, as this variable seems to capture much of the impact of leverage in returns."

parison, my model produces the relationship between stock returns, book-to-market values, and financial leverage, along with a negative distress premium. Second, Tables IV and V in Gomes and Schmid show that the market leverage premium between the top and bottom quintiles is 4.2 percent per year, whereas the book-to-market premium is 7 percent per year in the data. Their model generates 3 percent and 3.7 percent for these quantities, respectively, whereas my model generates 1.7 percent and 5.9 percent, respectively. This difference is in line with the intuition that book-to-market value has hardly any explanatory power above and beyond market leverage when there is no heterogeneity in cash-flow risk, which is the setup in Gomes and Schmid (2010). Third, an advantage of my approach is that I use regressions rather than double-sorting of portfolios according to book-to-market value and market leverage as in Gomes and Schmid (2010). Although the double sorting exercise in Gomes and Schmid shows that the market leverage premium in each book-to-market quintile is lower than the unconditional premium to market leverage, it does not tell us whether book-to-market value subsumes market leverage in explaining returns, because if we sorted the firms first according to market leverage and then by book-to-market value, we would also find that the book-to-market premium in each market leverage quintile was lower than the unconditional value premium.

7 Conclusion

This paper captures the following empirical regularities in a model where the risk-neutral and real default probabilities do not comove perfectly across the firms.

1- Firms with a higher likelihood of default have lower returns, as discussed in CHS (2009) and the references therein.

2- Firms with higher earnings-price ratios and higher book-to-market values have higher stock returns.

3- Firms with higher bond yields have higher stock returns.

4- Stock returns are positively related to market leverage but are insensitive to book leverage.

5- Stock returns are less sensitive to market leverage than to book-to-market value.

6- Market leverage is only weakly linked to stock returns after controlling for book-to-market value.

7- Book leverage remains insensitive to stock returns after controlling for book-to-market value, but becomes negatively related to stock returns after controlling for market leverage.

Aside from matching these regularities, the paper makes an additional claim that can be checked empirically, for example, by using the market data on credit default swaps: Firms with a higher default risk under the risk-neutral measure should have higher returns. Given that the CDS intruments are relatively new and currently do not cover the entire Compustat/CRSP universe, testing this hypothesis will be problematic with current data. I hope that we will be able to test this hypothesis in the near future. So far, the findings of Anginer and Yildizhan (2010) using bond yields seem to support this claim.

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9 Appendix

9.1 Determination of Market Values and Optimal Boundaries

Proposition 1 and the analysis in Section 3.1 suggest that we can separate the HJB equations for equity and debt into separate regions according to the value of y_0, $\bar{y}$, and y_B. There are three possible cases depending on the positioning of $\bar{y}$ relative to y_0 and y_B.

Case 1, $y_0 < \bar{y} < y_B$: The optimal policy of the firm suggests that we can separate the HJB equations for debt and equity into two separate equations, one for region $0 < y < \bar{y}$, and one for region $\bar{y} < y < y_B$. If we denote them region 1 and 2, respectively, and note that $y_0 < \bar{y}$, the equations for these regions are

$$
\begin{aligned}
(r-\mu+\lambda) E_1(y) &= (1-\tau)(1-y) - \mu y E_1'(y) + \frac{1}{2}\sigma^2 y^2 E_1''(y) \\
&\quad + \lambda \left(E_1(y_0) + (1-b) D_1(y_0) - \frac{y}{y_0} D_1(y_0) \right) \\
(r-\mu+\lambda) E_2(y) &= (1-\tau)(1-y) - \mu y E_2'(y) + \frac{1}{2}\sigma^2 y^2 E_2''(y) \\
(r-\mu+\lambda) D_1(y) &= y - \mu y D_1'(y) + \frac{1}{2}\sigma^2 y^2 D_1''(y) + \lambda \frac{y}{y_0} D_1(y_0) \\
(r-\mu+\lambda) D_2(y) &= y - \mu y D_2'(y) + \frac{1}{2}\sigma^2 y^2 D_2''(y) + \lambda \frac{\eta(1-\tau)}{r-\mu}
\end{aligned}
$$

with boundary conditions

$$\begin{aligned}
E_2(y_B) &= E_2'(y_B) = 0 \\
D_2(y_B) &= \frac{\eta(1-\tau)}{r-\mu} \\
E_1(\bar{y}) &= E_2(\bar{y}) \\
E_1'(\bar{y}) &= E_2'(\bar{y}) \\
D_1(\bar{y}) &= D_2(\bar{y}) \\
D_1'(\bar{y}) &= D_2'(\bar{y}),
\end{aligned}$$

where the last four conditions come from the fact that $y = \bar{y}$ is a transitional boundary.[15] Since the HJB equations are second order ordinary differential equations, their solution has the form

$$\begin{aligned}
E_1(y) &= \frac{1-\tau}{r-\mu+\lambda} + \frac{\lambda}{r-\mu+\lambda}\left(E_1(y_0) + (1-b)D_1(y_0)\right) - \left(\frac{1-\tau}{r+\lambda} + \frac{\lambda}{r+\lambda}\frac{D_1(y_0)}{y_0}\right)y \\
&\quad + A_1 y^{\beta_1} + A_2 y^{\beta_2} \\
E_2(y) &= \frac{1-\tau}{r-\mu+\lambda} - \frac{1-\tau}{r+\lambda}y + B_1 y^{\beta_1} + B_2 y^{\beta_2} \\
D_1(y) &= \left(1 + \lambda\frac{D_1(y_0)}{y_0}\right)\frac{y}{r+\lambda} + M_1 y^{\beta_1} + M_2 y^{\beta_2} \\
D_2(y) &= \frac{y}{r+\lambda} + \frac{\lambda}{r-\mu+\lambda}\frac{\eta(1-\tau)}{r-\mu} + N_1 y^{\beta_1} + N_2 y^{\beta_2},
\end{aligned}$$

where $\beta_1 < 0$ and $\beta_2 > 1$ are the roots of

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu + \frac{1}{2}\sigma^2\right)\beta - (r-\mu+\lambda) = 0,$$

because we need $r > \mu$ for convergence of the market values of debt and equity. This implies that we need to find two constants of integration for each of the $E_1(y)$, $E_2(y)$, $D_1(y)$, and $D_2(y)$, and also the values of y_B, y_0, and $\bar{y}$. We need 11 equations to solve for them. Seven of

[15] See Dixit (1993, p.30) for the details regarding transitional boundaries.

these come from the aforementioned boundary conditions and two more come from the fact that the option value of bankruptcy should not explode as $X \to \infty$ or equivalently $y \to 0^+$, which implies that $A_1 = M_1 = 0$. The last two come from the definitions of y_0 and $\bar{y}$, that is,

$$\begin{aligned} E_1'(y_0) + (1-b) D_1'(y_0) &= 0 \\ E_1(y_0) + (1-b) D_1(y_0) - \frac{\bar{y}}{y_0} D_1(y_0) &= 0. \end{aligned}$$

Case 2, $\bar{y} \geq y_B$: If $\bar{y} \geq y_B$, proposition 1 implies that the firm goes bankrupt only when y hits y_B and never at the time of debt maturity. Therefore, the firms are active only in the region $0 < y < y_B$. This is similar to region 1 of case 1 above, because the firm always refinances at the time of debt maturity. As a result, the HJB equations relevant for this case are

$$\begin{aligned} (r-\mu+\lambda) E_1(y) &= (1-\tau)(1-y) - \mu y E_1'(y) + \frac{1}{2}\sigma^2 y^2 E_1''(y) \\ &\quad + \lambda \left(E_1(y_0) + (1-b) D_1(y_0) - \frac{y}{y_0} D_1(y_0) \right) \\ (r-\mu+\lambda) D_1(y) &= y - \mu y D_1'(y) + \frac{1}{2}\sigma^2 y^2 D_1''(y) + \lambda \frac{y}{y_0} D_1(y_0) \end{aligned}$$

with boundary conditions

$$\begin{aligned} E_1(y_B) &= E_1'(y_B) = 0 \\ D_1(y_B) &= \frac{\eta(1-\tau)}{r-\mu}. \end{aligned}$$

Since the HJB equations are second order ordinary differential equations, their solution has the form

$$\begin{aligned} E_1(y) &= \frac{1-\tau}{r-\mu+\lambda} + \frac{\lambda}{r-\mu+\lambda}\left(E(y_0)+(1-b)D(y_0)\right) - \left(\frac{1-\tau}{r+\lambda} + \frac{\lambda}{r+\lambda}\frac{D_1(y_0)}{y_0}\right)y \\ &\quad + A_1 y^{\beta_1} + A_2 y^{\beta_2} \\ D_1(y) &= \left(1+\lambda\frac{D_1(y_0)}{y_0}\right)\frac{y}{r+\lambda} + M_1 y^{\beta_1} + M_2 y^{\beta_2}, \end{aligned}$$

where $\beta_1 < 0$ and $\beta_2 > 1$ are the roots of

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu + \frac{1}{2}\sigma^2\right)\beta - (r-\mu+\lambda) = 0.$$

This implies that we need to find two constants of integration for each of the $E_1(y)$ and $D_1(y)$, and also the values of y_B and y_0. We need six equations to solve for them. Three of these come from the aforementioned boundary conditions and two more come from the fact that the option value of bankruptcy should vanish as $X \to \infty$ or equivalently $y \to 0^+$, which implies that $A_1 = M_1 = 0$. The last one comes from the definition of y_0 and $\bar{y}$, that is,

$$E_1'(y_0) + (1-b)D_1'(y_0) = 0.$$

Case 3, $\bar{y} \le y_0$: The optimal policy of the firm suggests that we can separate the HJB equations for debt and equity into two separate equations, one for region $0 < y < \bar{y}$, and one for region $\bar{y} < y < y_B$. If we denote them region 1 and 2, respectively, and note that $y_0 \ge \bar{y}$, the equations for these regions are

$$
\begin{aligned}
(r-\mu+\lambda) E_1(y) &= (1-\tau)(1-y) - \mu y E_1'(y) + \frac{1}{2}\sigma^2 y^2 E_1''(y) \\
&\quad + \lambda\left(E_2(y_0) + (1-b) D_2(y_0) - \frac{y}{y_0} D_2(y_0)\right) \\
(r-\mu+\lambda) E_2(y) &= (1-\tau)(1-y) - \mu y E_2'(y) + \frac{1}{2}\sigma^2 y^2 E_2''(y) \\
(r-\mu+\lambda) D_1(y) &= y - \mu y D_1'(y) + \frac{1}{2}\sigma^2 y^2 D_1''(y) + \lambda \frac{y}{y_0} D_2(y_0) \\
(r-\mu+\lambda) D_2(y) &= y - \mu y D_2'(y) + \frac{1}{2}\sigma^2 y^2 D_2''(y) + \lambda \frac{\eta(1-\tau)}{r-\mu}
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
E_2(y_B) &= E_2'(y_B) = 0 \\
D_2(y_B) &= \frac{\eta(1-\tau)}{r-\mu} \\
E_1(\bar{y}) &= E_2(\bar{y}) \\
E_1'(\bar{y}) &= E_2'(\bar{y}) \\
D_1(\bar{y}) &= D_2(\bar{y}) \\
D_1'(\bar{y}) &= D_2'(\bar{y}),
\end{aligned}
$$

where the last four conditions come from the fact that $y = \bar{y}$ is a transitional boundary.[16] Since the HJB equations are second order ordinary differential equations, their solution has

[16] See Dixit (1993, p.30) for the details regarding transitional boundaries.

the form

$$
\begin{aligned}
E_1(y) &= \frac{1-\tau}{r-\mu+\lambda} + \frac{\lambda}{r-\mu+\lambda}\left(E_2(y_0) + (1-b)\,D_2(y_0)\right) - \left(\frac{1-\tau}{r+\lambda} + \frac{\lambda}{r+\lambda}\frac{D_2(y_0)}{y_0}\right) y \\
&\quad + A_1 y^{\beta_1} + A_2 y^{\beta_2} \\
E_2(y) &= \frac{1-\tau}{r-\mu+\lambda} - \frac{1-\tau}{r+\lambda} y + B_1 y^{\beta_1} + B_2 y^{\beta_2} \\
D_1(y) &= \left(1 + \lambda \frac{D_2(y_0)}{y_0}\right) \frac{y}{r+\lambda} + M_1 y^{\beta_1} + M_2 y^{\beta_2} \\
D_2(y) &= \frac{y}{r+\lambda} + \frac{\lambda}{r-\mu+\lambda}\frac{\eta(1-\tau)}{r-\mu} + N_1 y^{\beta_1} + N_2 y^{\beta_2},
\end{aligned}
$$

where $\beta_1 < 0$ and $\beta_2 > 1$ are the roots of

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu + \frac{1}{2}\sigma^2\right)\beta - (r-\mu+\lambda) = 0.$$

This implies that we need to find two constants of integration for each of the $E_1(y)$, $E_2(y)$, $D_1(y)$, and $D_2(y)$, and also the values of y_B, y_0 and $\bar{y}$. We need 11 equations to solve them. Seven of these come from the aforementioned boundary conditions and two more come from the fact that the option value of bankruptcy should not explode as $X \to \infty$ or equivalently $y \to 0^+$, which implies that $A_1 = M_1 = 0$. The last two come from the definitions of y_0 and $\bar{y}$, that is,

$$
\begin{aligned}
E_2'(y_0) + (1-b)\,D_2'(y_0) &= 0 \\
E_2(y_0) + (1-b)\,D_2(y_0) - \frac{\bar{y}}{y_0} D_2(y_0) &= 0.
\end{aligned}
$$

9.2 Proof of $y_0 < \bar{y} < y_B$ when $b = 0$ (Proposition 2)

The normalized net gain of restructuring to the shareholders is defined as

$$S_{N,b}(y) \equiv E(y_0) + (1-b)\,D(y_0) - \frac{y}{y_0} D(y_0) - E(y),$$

which is simply the net gain divided by cash flow after restructuring.

Lemma 4 *If $\bar{y} \geq y_B$, the normalized net gain in the absence of costs of debt issuance, $S_{N,0}(y)$, is decreasing in y for $y \geq y_0$.*

Proof. When $\bar{y} \geq y_B$, the active firms can only be in the first region. Therefore, we can write

$$S_{N,0}(y) = E_1(y_0) + D_1(y_0) - \frac{y}{y_0} D_1(y_0) - E_1(y),$$

which has the first and second derivatives

$$\begin{aligned} S'_{N,0}(y) &= -\left(\frac{D_1(y_0)}{y_0} + E'_1(y) \right) \\ S''_{N,0}(y) &= -E''_1(y). \end{aligned}$$

Since the term $A_2 y^{\beta_2}$ captures the value of the bankruptcy option, which is positive, we have $A_2 > 0$. Combining this with $\beta_2 > 1$ we get $E''_1(y) > 0$ and hence $S''_{N,0}(y) < 0$. Moreover, using $E'_1(y_0) + D'_1(y_0) = 0$ when $b = 0$, $\beta_2 > 1$, and $A_2 > 0$, it is straightforward to show that $M_2 < 0$ and

$$S'_{N,0}(y_0) = -\left(\frac{D_1(y_0)}{y_0} + D'_1(y_0) \right) < 0.$$

Combining $S'_{N,0}(y_0) < 0$ and $S''_{N,0}(y) < 0$ we get $S'_{N,0}(y) < 0$ for $y \geq y_0$. ■

This lemma combined with $y_B > y_0$ and $S_{N,0}(y_0) = 0$ leads to the following corollary:

Corollary 5 *If $\bar{y} \geq y_B$, $S_{N,0}(y_B) < 0$.*

Now we can show $y_0 < \bar{y} < y_B$ when $b = 0$. Let $S_0(y) = E(y_0) + D(y_0) - \frac{y}{y_0} D(y_0)$ be the value of shareholder surplus when the cost of issuing debt is zero. Since $S_0(y_0) = E(y_0) > 0$, $S'_0(y) < 0$ and $S_0(\bar{y}) = 0$ by definition, it immediately follows that $\bar{y} > y_0$.

Suppose $\bar{y} \geq y_B$. When $\bar{y} \geq y_B$ the firms are active only in the first region and we can write

$$S_0(y) = E_1(y_0) + D_1(y_0) - \frac{y}{y_0} D_1(y_0),$$

which implies that $S_0(y_B) > 0$ because $S_0'(y) < 0$ and $S_0(\bar{y}) = 0$. Moreover, using $E_1(y_B) = 0$ when $\bar{y} \geq y_B$, we get $S_{N,0}(y_B) > 0$. However, this contradicts the corollary shown above. Therefore, $\bar{y} < y_B$.

9.3 Long-Run Distribution of Firms

In order to have a stationary long-run distribution, I assume that when a firm goes bankrupt it is replaced by another firm with the same cash-flow riskiness, ρ, that chooses its initial capital structure optimally, which implies that $y = y_0$ at the inception of the new firm. Let $\Xi(z|\rho)$ be the the proportion of the firms with $y/y_B < e^z$ among the firms with cash-flow riskiness, ρ, and let $\xi(z|\rho)$ be the corresponding distribution function. If we define $z = \ln y/y_B$ and $z_0 = \ln y_0/y_B$, this distribution is the same as the long-run distribution of a process with an absorbing barrier at $z = 0$ and which is reset to $z = z_0$ at rate λ. The resetting of this process occurs this way because debt maturity occurs at rate λ and implies that either the firm returns to $y = y_0$ via refinancing or it goes bankrupt and is replaced by a firm with $y = y_0$. Therefore, until z hits the absorbing barrier it follows the process

$$dz = -\left(\mu_X - \frac{1}{2}\sigma^2\right)dt + \sigma dt + (z_0 - z)\,dN,$$

where

$$dN = \begin{cases} 1 \text{ with probability } \lambda dt \\ 0 \text{ with probability } 1 - \lambda dt \end{cases}.$$

Note that y_B and y_0 depend on ρ and hence a and a_0 also depend on ρ, which is different across firms.

Discretizing the process for z as in Dixit and Pindyck (1994, pp. 272-277), one can show that the distribution of the firms satisfies the following Kolmogorov backward equation

$$
\begin{aligned}
\frac{1}{2}\sigma^2\xi''(z) + \left(\mu_X - \frac{1}{2}\sigma^2\right)\xi'(z) - \lambda\xi(z) &= 0 \\
\xi(0) &= 0 \\
\frac{1}{2}\sigma^2\left[\xi'\left(z_0^+\right) - \xi'\left(z_0^-\right) - \xi'\left(0^-\right)\right] + \lambda &= 0 \\
\int_{-\infty}^{0} \xi(z)\, dz &= 1,
\end{aligned}
$$

where we drop ρ in $\xi(z|\rho)$ for simplicity. The solution to these equations is given by

$$
\xi(z) = \begin{cases} \bar{A}e^{\alpha_2 z} \text{ if } z \leq z_0 \\ \bar{B}\left(e^{\alpha_1 z} - e^{\alpha_2 z}\right) \text{ if } z_0 < z \leq 0 \end{cases},
$$

where $\alpha_1 < 0 < \alpha_2$ are the roots of

$$
\frac{1}{2}\sigma^2\alpha^2 + \left(\mu_X - \frac{1}{2}\sigma^2\right)\alpha - \lambda = 0
$$

and $\bar{A}$ and $\bar{B}$ satisfy

$$
\begin{aligned}
\bar{A}\alpha_2 e^{\alpha_2 z_0} - \left[\alpha_1\left(e^{\alpha_1 z_0} - 1\right) - \alpha_2\left(e^{\alpha_2 z_0} - 1\right)\right]\bar{B} &= \frac{2\lambda}{\sigma^2} \\
\frac{e^{\alpha_2 z_0}}{\alpha_2}\bar{A} - \left(\frac{e^{\alpha_1 z_0} - 1}{\alpha_1} - \frac{e^{\alpha_2 z_0} - 1}{\alpha_2}\right)\bar{B} &= 1.
\end{aligned}
$$

Once we find the distribution for the process a, finding the distribution of the firms in terms of y is straightforward via $\varphi(y|\rho) = \xi\left(\ln\left(y/y_B\right)|\rho\right)/y$.

Finally, the long-run distribution of the firms is given by

$$
f(y, \rho) = \varphi(y|\rho)\, h(\rho),
$$

where $h(\rho)$ is distribution of ρ's and is assumed to be uniform between ρ_L and ρ_H, and $\varphi(y|\rho)$ is the distribution of firms that have the riskiness of cash flows measured by ρ.

9.4 Distribution of Time to Bankruptcy

In this section I derive the moment generating function for the distribution of time to bankruptcy.

As we have done with the market value of equity, we divide the the state space into two regions. If we define $a = \ln y/y_B$, and $\bar{a} = \ln \bar{y}/y_B$, and $a_0 = \ln y_0/y_B$, then we have $a = 0$ as the absorbing barrier so that

$$da = mdt + \sigma dw + \left[\mathbb{I}_{a>\bar{a}}(0 - a) + (1 - \mathbb{I}_{a>\bar{a}})(a_0 - a)\right] dN,$$

where $m = -\left(\mu_X - \frac{1}{2}\sigma^2\right)$ and $\mathbb{I}_{a>\bar{a}}$ is the indicator function that is equal to 1 if $a > \bar{a}$ and 0 otherwise and

$$dN = \begin{cases} 1 \text{ with probability } \lambda dt \\ 0 \text{ with probability } 1 - \lambda dt \end{cases},$$

where being hit by the λ-shock when $a > \bar{a}$ implies that the firm goes bankrupt, which we denote as a jump to the absorbing barrier, and being hit by this shock when $a < \bar{a}$ implies resetting to a_0. Note that y_B, y_0, and $\bar{y}$ depend on ρ and hence a, $\bar{a}$, and a_0 also depend on ρ, which is different across firms.

Using this and denoting the regions $a < \bar{a}$ as region 1 and $a > \bar{a}$ as region 2, we find that the distribution of time to bankruptcy satisfies the following Kolmogorov forward equations

$$\begin{aligned} g_t^1(t,a) &= mg_a^1(t,a) + \frac{1}{2}\sigma^2 g_{aa}^1(t,a) + \lambda\left(g^1(t,a_0) - g^1(t,a)\right) \\ g_t^2(t,a) &= mg_a^2(t,a) + \frac{1}{2}\sigma^2 g_{aa}^2(t,a) + \lambda\left(\delta(t) - g^2(t,a)\right) \end{aligned}$$

subject to boundary conditions

$$\begin{aligned}
g^2(t,0) &= \delta(t) \\
g^1(0,a) &= g^2(0,a) = 0 \text{ for } a < 0 \\
\lim_{a \to -\infty} g_a^1(t,a) &= 0 \\
g^1(t,\bar{a}) &= g^2(t,\bar{a}) \\
g_a^1(t,\bar{a}) &= g_a^2(t,\bar{a}),
\end{aligned}$$

where $\delta(t)$ is the Dirac-Delta function and the last two conditions come from the fact that $\bar{a}$ is the transitional boundary.

If we define the Laplace transform as

$$\gamma(s,a) = \int_0^\infty e^{-st} g(t,a)\, dt,$$

we can reduce the Kolmogorov equations to the following second order ODEs

$$\begin{aligned}
(s+\lambda)\gamma^1(s,a) &= m\gamma_a^1(s,a) + \frac{1}{2}\sigma^2\gamma_{aa}^1(t,a) + \lambda\gamma^1(t,a_0) \\
(s+\lambda)\gamma^2(s,a) &= m\gamma_a^2(s,a) + \frac{1}{2}\sigma^2\gamma_{aa}^2(t,a) + \lambda,
\end{aligned}$$

subject to boundary conditions

$$\begin{aligned}
\gamma^2(s,0) &= 1 \\
\lim_{a \to -\infty} \gamma_a^1(s,a) &= 0 \\
\gamma^1(s,\bar{a}) &= \gamma^2(s,\bar{a}) \\
\gamma_a^1(s,\bar{a}) &= \gamma_a^2(s,\bar{a}),
\end{aligned}$$

which gives us the solution

$$\begin{aligned}\gamma^1(s,a) &= \tilde{A}\left(\frac{\lambda}{s}e^{\theta_2 a_0}+e^{\theta_2 a}\right)\\ \gamma^2(s,a) &= \frac{\lambda}{s+\lambda}+\frac{s}{s+\lambda}e^{\theta_2 a}+\tilde{B}\left(e^{\theta_1 a}-e^{\theta_2 a}\right),\end{aligned}$$

where $\theta_2 > 0 > \theta_1$ are the roots of

$$\frac{1}{2}\sigma^2\theta^2 + m\theta - (s+\lambda) = 0$$

and $\tilde{A}$ and $\tilde{B}$ satisfy

$$\begin{aligned}\left(\frac{\lambda}{s}e^{\theta_2 a_0}+e^{\theta_2 \bar{a}}\right)\tilde{A} &= \frac{\lambda}{s+\lambda}+\frac{s}{s+\lambda}e^{\theta_2 \bar{a}}+\tilde{B}\left(e^{\theta_1 \bar{a}}-e^{\theta_2 \bar{a}}\right)\\ \theta_2 e^{\theta_2 \bar{a}}\tilde{A} &= \frac{s}{s+\lambda}\theta_2 e^{\theta_2 \bar{a}}+\tilde{B}\left(\theta_1 e^{\theta_1 \bar{a}}-\theta_2 e^{\theta_2 \bar{a}}\right).\end{aligned}$$

Having found this, we can derive the moment generating function and cumulant generating function.

$$\begin{aligned}M(s,a) &= \gamma(-s,a)\\ K(s,a) &= \ln M(s,a),\end{aligned}$$

from which we can generate various distress measures the first of which is simply the mean time to bankruptcy given by $M'(0,a) = K'(0,a) = E[t|a]$.[17] The second and the third ones are saddlepoint approximations to the probability of default within one year using normal and inverse Gaussian bases.[18]

The steps for the saddlepoint approximation to the cumulative distribution function

[17] Here and in the following, the derivative with respect to the moment generating function and cumulant generating function is always taken with respect to s.

[18] See sections 1 and 16 in Butler (2007) for an excellent introduction to saddlepoint approximations and their properties. I choose the normal base because it is the standard base and the inverse Gaussian base because the probability density function of hitting time is an inverse Gaussian distribution when $\lambda = 0$.

(CDF) of first hitting time using the normal base are:[19]

1. Find $K'(s,a)$ and $K''(s,a)$.

2. Since we are interested in the probability of default within one year and our model is parameterized for a yearly frequency, find the values of $\hat{s}$, $\hat{w}$, and $\hat{u}$ that satisfy

$$\begin{aligned} K'(\hat{s},a) &= 1 \\ \hat{w} &= \operatorname{sgn}(1-K'(0,a))\sqrt{2\{\hat{s}x - K(\hat{s})\}} \\ \hat{u} &= \hat{s}\sqrt{K''(\hat{s})}. \end{aligned}$$

3. Find the probability of default within 1 year, using

$$\hat{F}(t) = \begin{cases} \Phi(\hat{w}) + \phi(\hat{w})(1/\hat{w} - 1/\hat{u}) & \text{if } K'(0,a) \neq 1 \\ \frac{1}{2} + \frac{K'''(0,a)}{6\sqrt{2\pi}K''(0,a)^{3/2}} & \text{if } K'(0,a) = 1 \end{cases}.$$

Note that $\hat{s}$ should be the root of $K'(s,a) = 1$ for which $M(s,a)$ converges. Since $K''(s,a) > 0$ in the region of convergence, this root is unique.

The steps for the saddlepoint approximation to the CDF of first hitting time using the inverse Gaussian base are:[20]

1. Due to the scale invariance property of the base used in the saddlepoint approximation, the family of inverse Gaussian bases can be reduced to a one-parameter family of inverse Gaussian bases where the base distribution is given by $z\tilde{~}IG(\alpha,\alpha^3)$ with the cumulant generating function

$$L(v) = \alpha^{-1} + \left(\alpha^{-2} - 2v\right)^{1/2},$$

and hence the choice of the suitable candidate from the IG family reduces to the choice

[19] See Butler (2007) Chapter 1. Note that the approximation to CDF does not require normalization, unlike the approximation to PDF.

[20] You will use the inverse Gaussian that comes from the problem of regular Brownian motion hitting times as the base. The following is adapted from Butler (2007, Ch. 16).

of suitable α.[21] The PDF of the chosen base distribution is called $\lambda(z)$ and its CDF is called $\Lambda(z)$.

2. Matching the square of the third standardized cumulants of the base distribution and the distribution to be approximated gives us the choice of a as

$$\hat{\alpha} = \frac{K'''(\hat{s})^2}{9K''(\hat{s})^3}\left(1 + \hat{w}\sqrt{\frac{K'''(\hat{s})^2}{9K''(\hat{s})^3}}\right)^{-1} \quad \text{if } K'''(\hat{s}) > 0,$$

where $\hat{s}$ and $\hat{w}$ are given above. Then we can find the saddlepoint approximation for the CDF of x

$$\hat{F}(t) = \begin{cases} \Lambda(\hat{z}) + \lambda(\hat{z})\left(1/\hat{v} - \sqrt{L''(\hat{v})}/\hat{u}\right) & \text{if } K'(0,a) \neq 1 \\ \hat{F}(K'(0,a)) & \text{if } K'(0,a) = 1 \end{cases},$$

where

$$\begin{aligned} \hat{z} &= \hat{\alpha} + \frac{\hat{\alpha}^2}{2}\left(\hat{w}^2 + \hat{w}\sqrt{\hat{w}^2 + 4/\hat{\alpha}}\right) \\ \hat{v} &= \frac{1}{2}\left(\hat{\alpha}^{-2} - \hat{z}^{-2}\right). \end{aligned}$$

$\hat{F}(K'(0,a))$ is given by equation (16.6) in Butler (2007). Here, it is implicitely assumed that $\hat{\alpha} > 0$, which holds for the parameterization here.[22]

3. For the case $K'''(\hat{s}) < 0$, we can use the formula given by Wood et al. (1993).

9.5 Simulated Portfolio Returns

[21] See Wood, Booth, and Butler (1993) and Butler (2007). This base can be derived by taking $y\tilde{\ }IG(\mu,\sigma^2)$ as the base distribution, rescaling the base variable as $z \equiv \sigma^2/\mu^3 * y\tilde{\ }IG\left(\sigma^2/\mu^2, \sigma^6/\mu^6\right)$, and replacing $\alpha \equiv \sigma^2/\mu^2$ to get an equivalent base.

[22] If this condition does not hold, Butler (2007) suggests the use of a small value for $\hat{\alpha}$, such as $\hat{\alpha} = 0.1$.

Table 5: **Simulation of equally weighted portfolios.** At the beginning of each year, stocks are ranked according to increasing price-earnings ratios, decreasing expected time to default, and increasing bond yields as a proxy for financial distress, under real and risk-neutral measures, increasing book-to-market values, market leverage, and book leverage to form equally weighted portfolios. A total of 1200 firms are simulated over 1200 months 100 times. The first 600 months are dropped in each simulation to allow the simulations to converge to the steady state. The table reports the time series means of equally weighted portfolio returns averaged across simulations.

Portfolio Returns with Different Rankings					
A. Cash-Flow Model					
Portfolio	1	2	3	4	5
Distress	13.74	12.66	11.47	10.24	10.15
Risk-Neutral Distress	8.84	10.24	11.52	12.78	14.89
Earnings/Price	8.84	10.25	11.52	12.77	14.90
Bond Yield	10.42	10.97	11.41	11.83	13.63
B. Investment Model					
Portfolio	1	2	3	4	5
Book-to-Market	8.85	10.24	11.52	12.77	14.89
Market Leverage	11.17	11.38	11.21	11.31	13.19
Book Leverage	12.27	11.99	11.34	10.75	11.91