Tobin's q, Economic Rents, and the Optimal Stock of Capital

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Abstract

Within optimal investment programs, the accumulation of capital is a stable function of marginal q. Much of the interest in q, however, derives from its potential to reflect the demand for capital when the optimal program changes. If the marginal return on capital diminishes as capital increases, the correspondence between marginal q and the optimal stock of capital can shift whenever investors alter their assessments of prospective economic rents. At such times, marginal q even could rise as the optimal stock of capital falls. In general, robust investment functions express optimal investment in terms of those variables that determine marginal q, rather than marginal q itself. However, under some restrictions (e.g. price-taking enterprises), marginal q may be sufficient to determine the optimal accumulation of capital even as the program changes. The conditions that make marginal q a sufficient statistic also make q a sufficient statistic.

Tobin's q, Economic Rents, and the Optimal Stock of Capital

The optimal stock of capital at any moment is an element of an intertemporal program of capital accumulation (Fisher 1930, Hicks 1946, Hirshleifer 1970, Hayashi 1982, Abel and Eberly 1993). Within this program, the rate of capital accumulation ordinarily can be expressed as a function of marginal q. The spirit of q, however, extends beyond the letter of the theory. Might the rate of accumulation of capital also be a function of marginal q at times when the optimal program shifts? Much of the practical interest in q rests on its ability to represent changes in the optimal accumulation of capital at moments when investors change their assessments of prospective returns (Keynes 1936 ch. 12, Tobin 1969 and 1982, Summers 1981, Hayashi 1982, Abel and Blanchard 1986).

This paper examines the relationship between marginal q and the optimal stock of capital for an enterprise whose prospective rate of return varies with its stock of capital.[1] Because both q and the optimal stock of capital are endogenous variables that depend on economic rents in these circumstances, the relationship between q and investment is prone to change when investors alter their assessments of prospective rents (Haavelmo 1944, Duesenberry 1948). At these times, q theory yields stable investment functions by expressing the optimal stock of capital either in terms of the returns to capital or in terms of the variables that determine these returns, rather than marginal q itself. If, however, the

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[1]If the distribution of returns is independent of the stock of capital, Abel and Eberly (1993) derive an investment function governed by q, the shadow price of capital, when investors must contend with adjustment costs, irreversibility, and uncertainty. Regarding adjustment costs and uncertainty, see for example, Witte (1963), Eisner and Strotz (1963), Lucas and Prescott (1971), Hayashi (1982), and Abel (1985). Arrow (1968), Myers (1977), McDonald and Siegel (1986), Pindyck (1988, 1991), and Greenwald and Stiglitz (1989) discuss the value of the option of waiting to invest. Wildasin (1984) and Hayashi and Inoue (1990) discuss the complications of multiple capital inputs. Imperfections in capital markets are discussed by Myers (1984), Myers and Majluf (1984), Chirinko (1987) and Fazzari et al. (1988); see also Hayashi (1985).

admissible functions for marginal q were restricted so that they differed by only one free parameter, then marginal q measured at any stock of capital would be sufficient to determine the optimal stock of capital. In these cases q also would be a sufficient statistic.

The first section of this paper describes the relationship between the optimal stock of capital and marginal q for an enterprise with one factor of production, capital, for which returns decline with increasing scale. When investors change their assessments of returns on capital, reducing (increasing) marginal q at the formerly optimal stock of capital, the optimal stock of capital also falls (rises), but the change in marginal q does not necessarily correspond closely to the magnitude of the change in the optimal stock of capital.

In the second section, q is a function of the stock of capital and leverage. In this case, a change in prospective rents may increase the demand for capital even though it reduces marginal q evaluated at the prevailing choice of capital and leverage, provided the optimal degree of leverage changes sufficiently. The third section comprises two numerical examples. The first illustrates the possibility that marginal q may fall below unity while the optimal stock of capital increases when investors revise their assessments of returns. The second shows how constraints on shareholders' access to equity financing may enforce a tendency for marginal q and capital to change in the same direction.

The potential for marginal q and the optimal stock of capital seemingly to change in opposite directions does not rest solely on conditions that violate the Modigliani-Miller theorem. If, in addition to capital, q is a function of at least one other variable that is controlled by investors (such as the employment of labor), then a change in prospective rents may increase the optimal stock of capital while it reduces marginal q, provided investors' optimal choices for these other variables change sufficiently due to the reassessment of rents.

I. q Depends Only on the Stock of Capital

The expected net revenue accruing to an enterprise depends on its stock of capital, its only factor of production. Capital does not depreciate, the price of a unit of capital is unity, and purchases of capital entail no adjustment costs. Given its capital, K, the enterprise produces $\tilde{Q}(K)$ units of output each period. The enterprise receives net revenue of $\tilde{P}(\tilde{Q}(K))$ for each unit of its output. Total expected net revenue each period is

$$(1)\quad REV(K) = \mathrm{E}[\ \tilde{P}(\tilde{Q}(K))\ \tilde{Q}(K)\ ]$$
$$REV^{\prime} > 0 \quad \text{and} \quad REV^{\prime\prime} < 0\ .$$

The rate of return on marginal units of capital ($REV^{\prime}$) falls as K increases because of diminishing returns, upward-sloping supply schedules for raw materials, or downward-sloping demand schedules for output. The expected rate of return each period is

$$(2)\quad r(K) = \frac{REV(K)}{K}\ .$$

Tobin's q is the ratio of the value of this enterprise to the replacement value of its capital. Assuming the conditions of the Modigliani-Miller theorem prevail, the value of the enterprise is the present value of its net revenues, discounted by its cost of capital, $\rho(K)$, which reflects the risks inherent in these revenues.[2]

$$(3a)\quad q(K_o) = \frac{\sum_{i=1}^{\infty}[\ REV(K_o)\ (1+\rho(K_o))^{-i}\ ]}{K_o} = \frac{r(K_o)}{\rho(K_o)} = 1 + \frac{r(K_o) - \rho(K_o)}{\rho(K_o)}\ .$$

[2] Modigliani and Miller (1958 and 1963), Miller and Modigliani (1961), Miller (1977), Lintner (1965). The cost of capital may vary with the stock of capital if the enterprise's beta varies with the enterprise's rents (Lang and Stultz 1993).

q reflects the average expected rent per unit of capital, $r(K)-\rho(K)$, for any fixed stock of capital. This static measure of q is not observed unless the prevailing stock of capital equals the optimal stock of capital. For example, if K is less than optimal, the value of the enterprise will reflect both the rents accruing to K and, in anticipation of the expansion of the enterprise, the present value of the rents accruing on the additional capital. Assuming that this additional investment is imminent, that K^o denotes the optimal stock of capital, and, for simplicity, that ρ is constant, the observed measure of q is

$$(3b)\quad q^{obs}(K_o) = q(K_o) + \frac{\int_{K_o}^{K^*} (REV'(K) - \rho)\, dK}{\rho\, K_o} .$$

Marginal q is the change in the value of the enterprise resulting from the addition of another unit of capital divided by the replacement value of this capital ($\Delta(qK)/\Delta K$):

$$(4)\quad \begin{aligned} \upsilon(K_o) &\equiv D_K(qK_o) = q + K_o\, D_K q = (REV' - qK_o\, \rho')\, /\, \rho\,, \quad \text{implying} \\ q(K_o) &= \int_0^{K_o} \upsilon(s)\, ds\, /\, K_o\,. \end{aligned}$$

q is the average of the marginal q for each unit of capital. This definition uses q rather than q^{obs}, because q^{obs} already anticipates the additional rents entailed by the shift to the optimal stock of capital.

If investors choose K to maximize their expected wealth, then marginal q, as defined in (4), equals unity for the optimal steady-state stock of capital (Fisher 1930, Hicks 1946, Hirshleifer 1970, and Hayashi 1982). Accordingly, the expected marginal rate of return equals the marginal cost of capital,

(5) $REV' = \rho + qK\rho'$.

Marginal q and the Optimal Stock of Capital: the One-Parameter Case

Suppose the cost of capital is constant, the production function is fixed, and the demand curve for the enterprise's product is known except for a proportional shift parameter, β. Then, from (4)

$$
(6) \quad \begin{aligned} \upsilon(K) &= \beta\,[P(K)\,Q(K)]' \,/\, \rho \\ q(K) &= \beta\,P(K)\,Q(K) \,/\, (\rho K)\,. \end{aligned}
$$

The functions P and Q fix the shape of the function for marginal q, while β determines the position of the function. The familiar price-taking enterprise is a special case of this example. Each value of β determines a unique value of K for which marginal q equals unity; consequently, the optimal stock of capital may be expressed as a function of β. Because β, in turn, can be determined uniquely from the value of q or marginal q at any K, the optimal stock of capital also can be expressed as a function of either q or marginal q for any K.

In general, when only one free parameter distinguishes the admissible, nonintersecting functions for marginal q, then the value of either marginal q or q for any K is sufficient to determine the value of this parameter and the optimal stock of capital. If two free parameters distinguish these functions (i.e. υ is always a straight line), then the values of both marginal q and q for any K would suffice to determine the optimal stock of capital.

Marginal q and the Optimal Stock of Capital: the General Case

If the shapes of demand curves, supply curves, or production functions change when investors revise their assessments of returns, then the function for marginal q may depend on more than one or two free parameters. In this case, q and marginal q are not sufficient for determining the optimal stock of capital.

Suppose the demand curve for output shifts upward and changes its shape (figure 1). Consider two possible shapes for the new demand curve that displace the function υ to either υ_{I} or υ_{II}. Both alternatives increase marginal q by the same amount at the prevailing stock of capital, K_o. This increase reflects only the shift of the function for marginal q at the existing stock of capital, not the slope of the function at that point or the rate of change of this slope as K increases. Because the increase in the optimal stock of capital depends on the slope of υ, and because υ_{I} ultimately falls more rapidly than υ_{II} as K increases, the optimal stock of capital increases more for the second alternative than it does for the first. Although q^{obs} reflects the total additional rent accruing to optimal investment, it does not necessarily reflect the rate of decay of rents as the stock of capital expands. In this example, q^{obs} and q do not reveal that the optimal stock of capital is greater for the second alternative, because the value of the additional rents for the first alternative exceeds that for the second (the area of region A exceeds that of B).

Proposition 1

If an enterprise earns rents, q varies only with the stock of capital, the discount rate is independent of K, and investors maximize their expected aggregate wealth,

then whenever marginal q is a sufficient statistic for determining the optimal stock of capital, q also is a sufficient statistic:

(i) Marginal q equals unity at the optimal stock of capital.

(ii) If one free parameter distinguishes the admissible functions for marginal q, the optimal stock of capital is a stable function of either marginal q or q.

(iii) In general, however, any reassessment of rents that increases (decreases) marginal q also increases (decreases) the optimal K, but the values of marginal q and q at the existing stock of capital are not sufficient for determining the optimal K.

(iv) If investment entails adjustment costs, marginal q and q are not sufficient to determine the rate of investment, because the optimal accumulation of capital depends on the shape of the function for marginal q.

The Rate of Accumulation of Capital: the Cost of Capital Is Independent of K

If investment entails adjustment costs that increase with the rate of accumulation of capital, then the optimal rate of investment depends on how rapidly rents erode as K rises. Suppose the average premium for acquiring new capital in any period, $C(\Delta K) \geq 0$, rises with investment, ΔK. The value of increasing investment one unit in the first period is:

$$\begin{aligned} &D_{\Delta K}\big(q(K_o+\Delta K)\,(K_o+\Delta K) - C(\Delta K)\,\Delta K \big) = \upsilon(K_o+\Delta K) - \alpha(\Delta K), \quad \text{where} \\ (7a) \qquad &\alpha(\Delta K) = C'(\Delta K)\,\Delta K + C(\Delta K)\,. \end{aligned}$$

In Figure 2, each period's investment corresponds to those points where υ-α equals unity. If investment were greater, the net present value of the expected returns to marginal investment in any period would be less than the steady-state price of capital. The accumulation of capital initially is greater in the upper panel even though the optimal stock of capital increases more in the lower panel, because rents erode more rapidly in the lower panel.

These sequences show the maximal economic rate of investment, not necessarily the optimal rate. Because adjustment costs increase with investment, if the accumulation of capital is concentrated too greatly in the earliest periods, then postponing some of this investment until later periods reduces total adjustment costs more than it reduces the present value of earnings. For example, postponing a unit of investment from the first to the second period would reduce adjustment costs by $\alpha(K_1$-$K_0)$ in the first period; in the second, adjustment costs would increase by $\alpha(K_2\text{-}K_1)$ and earnings in the first period would be reduced by $REV'(K_1)$. The value of delaying a unit of investment to period j is

$$(7b) \qquad \alpha(K_1-K_0) - \alpha(K_j-K_{j-1})\ (1+\rho)^{-(j-1)} - \sum_{i=1}^{j-1} REV'(K_i)\ (1+\rho)^{-i} \ .$$

If the function REV' were sufficiently concave (upper panel of the figure), the discrepancies between the earliest and latest $\alpha(\Delta K_j)$ would be relatively great, and the benefit of shifting investment into the last periods would exceed the opportunity cost.

Because optimal investment depends on the shape of υ as well as adjustment costs, investment could fall when marginal q increases if υ shifts from a sufficiently concave to a sufficiently convex profile. Suppose an enterprise had begun a program of substantial investment after υ had shifted upward and become very concave. If this function subsequently shifts up once more, but becomes sufficiently convex, the optimal rate of investment could fall even though marginal q increased at the prevailing stock of capital. Just as marginal q at any K is not sufficient to determine the change in the optimal stock of capital, it also is not sufficient to determine the timing of investment, unless restrictions allow the value of marginal q at a point to determine the entire function -- i.e., if υ had only one free parameter.

II. q Depends on the Stock of Capital and Leverage

Consider a one-period investment financed partly by shareholders (ϕ), partly by creditors (1-ϕ). All shareholders assess the uncertain rate of return on capital by the same probability distribution, $\text{pdf}_s(\tilde{r}\,|K)$. The corresponding distribution for creditors is $\text{pdf}_c(\tilde{r}\,|K)$. Because shareholders regard the enterprise's prospects more favorably than creditors, the shareholders' expected value of the rate of return on capital, $r_s(K)$, exceeds that of creditors, $r_c(K)$. The discount rate for shareholders and creditors alike is ρ, which does not vary with ϕ or K.

The rate of interest on debt, i, equates the creditors' expected return with their discount rate. The yield on debt is i provided the return is sufficiently great to pay creditors' claims, $(1+\tilde{r}) \geq (1-\phi)(1+i)$ or $\tilde{r} \geq i(1-\phi)-\phi$. Otherwise, the yield is only $(\tilde{r}+\phi)/(1-\phi)$. Equating the discount rate with the creditors' expected yield,

$$(8)\quad \rho = \int_{(1-\phi)i-\phi}^{\infty} i\ pdf_c(\tilde{r})d\tilde{r} + \int_{-100\%}^{(1-\phi)i-\phi} [(\tilde{r}+\phi)/(1-\phi)]\ pdf_c(\tilde{r})d\tilde{r}, \quad \text{or}$$

$$(9)\quad (1-\phi) = \int_{(1-\phi)i-\phi}^{\infty} i(1-\phi)/\rho\ pdf_c(\tilde{r})d\tilde{r} + \int_{-100\%}^{(1-\phi)i-\phi} (\tilde{r}+\phi)/\rho\ pdf_c(\tilde{r})d\tilde{r}.$$

The "risk premium," i/ρ, compensates creditors for the expected value of their losses so that the present value of their expected receipts equals the value of their investment in the enterprise.

Heterogeneous Expectations and q

q equals the sum of the values of equity and debt divided by the replacement value of capital goods. The value of debt is the present value of creditors' expected receipts, $K(1-\phi)$. Shareholders receive a payment only when creditors' claims are paid in full ($\tilde{r} \geq i(1-\phi)-\phi$).

$$(10)\quad q = \{ K \int_{(1-\phi)i-\phi}^{\infty} [\tilde{r}+\phi-(1-\phi)i]\ pdf_s(\tilde{r})d\tilde{r}\ /\ (1+\rho) \ + \ K(1-\phi)\}\ /\ K$$

$$= (1+r_s)/(1+\rho) \ + \ \{ \int_{(1-\phi)i-\phi}^{\infty} (1-\phi)i\ (pdf_c(\tilde{r}) - pdf_s(\tilde{r}))\ d\tilde{r}$$

$$+ \int_{-100\%}^{(1-\phi)i-\phi} (\tilde{r}+\phi)(pdf_c(\tilde{r}) - pdf_s(\tilde{r}))\ d\tilde{r}\ \}\ /\ (1+\rho)\ .$$

If pdf_c were identical to pdf_s or if ϕ were unity, then (10) would correspond to (3a). Using (8) and (9) the last equality of (10) can be rewritten:

$$(11)\quad q = (1+r_s)/(1+\rho) \ + \ \{ \rho(1-\phi) - \int_{(1-\phi)i-\phi}^{\infty} (1-\phi)i\ pdf_s(\tilde{r})d\tilde{r}$$

$$- \int_{-100\%}^{(1-\phi)i-\phi} (\tilde{r}+\phi)\ pdf_s(\tilde{r})d\tilde{r}\ \}\ /\ (1+\rho)\ .$$

Suppose that, for any K, pdf_s matches pdf_c in all respects, except that the shareholders' expectation of the return on capital exceeds that of creditors. Therefore, for given ϕ and K, the rate of interest that equates the expected return on debt with the discount rate is greater for creditors than it would be if shareholders were purchasing the debt. Denoting this difference by δ, from the shareholders' viewpoint (rewriting (8)):

$$(12)\quad \rho(1-\phi) = \left\{ \int_{(1-\phi)(i-\delta)-\phi}^{\infty} (1-\phi)(i-\delta)\, pdf_s(\tilde{r})d\tilde{r} + \int_{-100\%}^{(1-\phi)(i-\delta)-\phi} (\tilde{r}+\phi)\, pdf_s(\tilde{r})d\tilde{r} \right\}$$

$$= \left\{ \int_{(1-\phi)i-\phi}^{\infty} (1-\phi)i\, pdf_s(\tilde{r})d\tilde{r} + \int_{-100\%}^{(1-\phi)i-\phi} (\tilde{r}+\phi)\, pdf_s(\tilde{r})d\tilde{r} \right\}$$

$$- \int_{(1-\phi)(i-\delta)-\phi}^{\infty} (1-\phi)\delta\, pdf_s(\tilde{r})d\tilde{r} - \int_{(1-\phi)(i-\delta)-\phi}^{(1-\phi)i-\phi} (\tilde{r}+\phi-(1-\phi)i)\, pdf_s(\tilde{r})d\tilde{r}\,.$$

Substituting (12) for $\rho(1-\phi)$ in (11),

$$(13)\quad q = (1+r_s)/(1+\rho) - \left\{ (1-\phi)\delta \int_{(1-\phi)(i-\delta)-\phi}^{\infty} pdf_s(\tilde{r})d\tilde{r} \right.$$

$$\left. + \int_{(1-\phi)(i-\delta)-\phi}^{(1-\phi)i-\phi} (\tilde{r}+\phi-(1-\phi)i)\, pdf_s(\tilde{r})d\tilde{r} \right\} / (1+\rho)$$

$$= (1+r_s)/(1+\rho) - \lambda(\phi,K).$$

λ, a positive function which increases with leverage or K, reflects the expected value of the revenues forgone by shareholders in order to compensate the less optimistic creditors.

Asymmetric Taxation and q

The net return on the enterprise's stock of capital goods tends to rise with leverage when the enterprise pays a tax on returns that are distributed to shareholders, while paying no tax on returns distributed to creditors (King 1977; Auerbach 1979, 1983; DeAngelo and Masulis 1980; Poterba and Summers 1983). Assuming the return to shareholders is taxed as corporate income at rate τ when this return is positive, then (13) becomes

$$(14) \quad q = (1+r_s)/(1+\rho) - \lambda(\phi,K) - \tau \int_{(1-\phi)i}^{\infty} (\tilde{r}-(1-\phi)i)\, pdf_s(\tilde{r})d\tilde{r} \,/\, (1+\rho)$$

$$= (1+r_s)/(1+\rho) - \lambda(\phi,K) - \tau\ \mu(\phi,K)\ .$$

μ, a positive function, reflects the expected value of shareholders' returns subject to corporate income taxation. This tax liability decreases as either leverage or K increases.

Marginal q and the Optimal Choice of Capital and Leverage

The enterprise's shareholders maximize their wealth by maximizing q subject to the constraint that marginal q equals unity. Holding leverage constant, an investment of ΔK, costing shareholders $\phi\Delta K$, increases the value of shareholders' equity by $\Delta(qK)$-$(1$-$\phi)\Delta K$. The net change in shareholders' wealth is $\Delta(qK)$-ΔK, which is zero when υ equals unity. Therefore, selecting the optimal K and ϕ may be separated into two steps. First, for any value of ϕ, define $K^o(\phi)$, the value of K for which marginal q equals unity (see Figure 3). Second, maximize q with respect to ϕ, subject to the constraint that K equals $K^o(\phi)$: ϕ maximizes $q(\phi,K^o(\phi))$ at the point where K^o is tangent to a contour of q.

Given ϕ, applying (4) to (14) describes marginal q,

$$(15) \quad \upsilon = (1 + r_s + K\, r_s^{\prime})/(1+\rho) - (\lambda + K\, D_K\lambda) - \tau(\mu + K\, D_K\mu)\ .$$

Setting marginal q equal to unity, $K^o(\phi)$ is implicitly defined by

$$(16) \quad REV_s^{\prime} = \rho + (1+\rho)\{\ \tau(\mu + K\, D_K\mu) + (\lambda + K\, D_K\lambda)\ \}\ .$$

The expression on the right side of (16) defines the marginal cost of capital. If the premium that shareholders pay creditors (reflected in λ) is sufficiently great or rises sufficiently rapidly as K increases, then the shareholders' expected marginal return on capital exceeds ρ at the optimal stock of capital and the marginal cost of capital increases with K.[3]

The slope of K^o, which is zero only at isolated points, can be positive or negative (see appendix). When ϕ is near unity, the slope of K^o may be negative: an increase in equity financing may increase returns subject to taxation more than it diminishes the additional interest that shareholders must pay to creditors. Conversely, for values of ϕ nearer zero the slope of K^o may be positive: greater equity financing may increase the tax burden less than it diminishes the rate of interest on debt.

When the function K^o is tangent to a contour of q, their slopes are equal, implying

$$(17)\quad \frac{D_\phi \upsilon}{D_K \upsilon} = \frac{D_\phi q}{D_K q} .$$

When the enterprise earns economic rents (q exceeds unity), $D_\phi q$ ordinarily is not zero when $D_\phi v$ is zero (see appendix). Therefore, the slopes of K^o and the contour of q ordinarily are not zero at their point of tangency. Accordingly, the choice of ϕ does not necessarily minimize the effective cost of capital -- maximize q given K.

Proposition 2

If q is a function of leverage as described in (14), the shareholders' ability to invest in the enterprise is not constrained, shareholders maximize their aggregate wealth, and the discount rate for all investors is identical and constant,

then:

(i) marginal q equals unity at the optimal stock of capital;

(ii) both the marginal return and the marginal cost of capital exceed the discount rate;

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[3] λ, $D_K\lambda$ and μ are not negative; $D_K\mu$ is not positive. Consequently, the marginal cost of capital is less than ρ when the magnitude of $\tau K D_K\mu$ exceeds the sum of the other terms in the braces. That ρ is less than REV' when corporate income is taxed is a familiar result: $\mu + KD_K\mu$ ordinarily exceeds zero.

(iii) given K, the optimal choice of leverage ordinarily does not minimize the cost of capital;
(iv) the optimal K varies with leverage; and
(v) with a reassessment of rents, the change in optimal K may vary inversely with the change in marginal q if, for example, the shape of REV_s should become less concave as it shifts (the returns on inframarginal capital fall relative to the return on marginal capital).

The optimal stock of capital depends on the shape of the shape of REV_s. In Figure 4, a shift of REV_s displaces the graph of K^o downward from point 1; consequently marginal q at this point falls below unity. But the optimal value of K rises, because the slopes of K^o and the contours of q become steeper, displacing the tangency to point 2. From the total differential of (14), the slope along a contour of q is

$$(18)\quad D_\phi K|_{q=const} = \frac{D_\varphi(\tau\mu + \lambda)}{(REV_s^{\prime}-r_s)/(K(1+\rho)) - D_K(\tau\mu + \lambda)},$$

and (from (16)) the slope of K^o is

$$(19)\quad D_\phi K^o = \frac{D_\varphi\,(\,\tau(\mu + KD_K\mu) + (\lambda + KD_K\lambda)\,)}{REV_s^{\prime\prime} - D_K\,(\,\tau(\mu + KD_K\mu) + (\lambda + KD_K\lambda)\,)}.$$

As REV_s becomes less concave, both $REV_s^{\prime} - r_s$ and $REV_s^{\prime\prime}$ become less negative. Then, as shown in the figure (and in the next section's example), the slopes of K^o and the contours of q will tend to increase as the magnitudes of the denominators of (18) and (19) diminish.

III. Heterogeneous Investors and the Correspondence between Marginal q and the Optimal Stock of Capital

This section comprises two numerical examples. According to proposition 2, marginal q (evaluated at the existing stock of capital) and the demand for capital goods can change in opposite directions at the time an unanticipated event shifts the profile of expected rents. This section's first example illustrates this possibility. If, however, shareholders' endowments constrain their equity investments, then marginal q and the optimal stock of capital may tend to change in the same direction. This section's second example illustrates this conclusion. Nevertheless, in this second case, as in propositions 1 and 2, marginal q is not a sufficient statistic for the optimal stock of capital.

The Supply of Equity Financing Is Not Constrained

For shareholders, the initial distribution of the project's rate of return on capital, pdf_s, is rectangular: the range is 200 percentage points, and the mean equals

$$(20) \quad r_s(K) = 4. - 7.\times 10^{-13} K^4 .$$

This function implies that shareholders expect a marginal rate of return of 21.2 percent when K equals 1020. The function slopes downward steeply -- the inframarginal returns are much greater than the marginal return. For creditors, the distribution pdf_c is the same as that for shareholders except that its mean is lower and varies less with the stock of capital.

$$(21) \quad r_c(K) = r_s(K) - .9\,(r_s(K) - \rho) .$$

When K equals 1020 creditors expect a marginal rate of return on capital of 6.6 percent. The remaining parameters of (14) and (16) are

$$(22) \quad \begin{aligned} \rho &= .05 \\ \tau &= .5 \quad . \end{aligned}$$

Under the conditions stated in proposition 2, the optimal choice of (ϕ,K) is point 1 in Figure 4 and in Table 1. The marginal return on capital and the marginal cost of capital exceed both the investors' discount rate (5 percent) and the interest rate on debt, because the marginal rate of change of interest expense, $D_K\lambda$, is relatively great (see (16)). For this reason too, the rate of interest on debt, after corporate taxes, is less than the discount rate.

Suppose investors revise their assessments of the enterprise's prospective returns so that *REV* becomes less concave and marginal q falls below unity at point 1. For shareholders, the new pdf_s is identical to the first except that the mean becomes

$$(23)\quad r_s(K) = .5 - 9.157\times10^{-2}\ K^{.1}\ .$$

When K equals 1020 shareholders expect a marginal rate of return of 29.9 percent, almost 9 percentage points higher than before. But the function is now much flatter -- the returns on inframarginal capital are only modestly greater than that on marginal capital. The new pdf_c is defined by (21) and (23). Accordingly, when K equals 1020 creditors expect a marginal rate of return on capital of 7.5 percent, almost one percentage point higher than before.

With this revision, q falls substantially at point 1, and marginal q is less than unity. Although the marginal cost of funds increases more than the marginal return on capital at the formerly optimal choice of (ϕ,K) (Table 1, column 2), the optimal stock of capital increases. The new optimal choice of (ϕ,K) is point 2 in Figure 4 and in Table 1.

The expected rents on inframarginal investments fall substantially with the revision of expected returns, and creditors require a greater rate of interest to prevent their expected rate of return from falling (column 2 versus column 1). With the revised returns, interest expense also rises at a greater rate as leverage increases -- $D_\phi\lambda$ becomes more negative and

$D_{\phi}q$ increases. For these reasons, both leverage and the interest rate on debt are lower at point 2 than they are at point 1. Because the terms of credit have deteriorated, the hurdle rate required of investments also increases -- both λ and $D_K\lambda$ increase, raising the marginal cost of capital in (16). Accordingly, the marginal cost of capital, the marginal return on capital, and the rate of interest are greater at point 2 than at point 1. Given K, the choice of ϕ does not minimize the cost of capital at either point 1 or point 2. In both cases, a small reduction in leverage would reduce the premium paid to creditors more than it would increase the tax burden on the return to capital.

The Supply of Equity Financing Is Constrained

If shareholders possess only $510, so that their equity investment absorbs all their funds at point 1, then they are unable to provide $1,268 of equity in order to reach point 2. Their best strategy, under these circumstances, selects the ϕ that maximizes the value of their $510 equity investment (Figure 5).

With this equity constraint, the function $K^1(\phi)$ shows the enterprise's maximum stock of capital for each value of ϕ. Shareholders select the point on K^1 that maximizes the net worth of the enterprise,

$$(24) \quad V(\phi,K) \equiv qK - (1-\phi)K.$$

When equity constraints are binding, the optimal choice of K (point 3) lies on the section of K^1 beneath its intersection with $K_{\Pi}{}^{\circ}$; above this intersection, marginal q is less than unity. Marginal q exceeds unity at the optimal choice of K, and the marginal return on capital exceeds the marginal cost of capital.

Any shift of expected returns that reduces marginal q below unity (at the formerly optimal value of K) must also reduce the optimal stock of capital when equity constraints are binding. If the value of marginal q were greater than unity initially, then a shift of expected returns that reduces marginal q, but leaves it above unity at the formerly optimal value of K, may not reduce the optimal stock of capital.

IV. Conclusion

The relationship between q and the optimal stock of capital depends on the frequency with which investors alter their assessments of prospective economic rents. Although the rate of investment may be expressed as a function of marginal q within an optimal program, marginal q is not necessarily sufficient to determine the optimal accumulation of capital when investors alter their assessments of rents, thereby shifting the program. According to q theory, if investors often revise their assessments, stable investment functions would express the optimal accumulation of capital, not in terms of marginal q or q, but in terms of those variables that jointly determine q and the optimal stock of capital. If, however, restrictions on net demand curves and production functions restricted, in turn, the admissible functions for marginal q so that they differed by only one free parameter, then either q or marginal q measured at any stock of capital would be sufficient to determine the optimal accumulation of capital. Price-taking enterprises are an example of this restriction.

The possibility that marginal q and the accumulation of capital seemingly may move in opposite directions when the program shifts does not arise solely from the failure of the Modigliani-Miller theorem. They also may appear to be in opposition when, in addition to the stock of capital, an enterprise's rents depend on variables such as the employment of labor, the choice of technology, or the mix of outputs. For instance, when q depends on the employment of labor, the stock of capital for which marginal q equals unity also varies with labor services (see the appendix). An unanticipated reduction in personal income taxes may reduce the wage rate employers pay, thereby increasing profits and marginal q, other things equal. But the optimal stock of capital could fall (or at least increase comparatively little) to the degree the lower wage induces the substitution of labor for capital.

Appendix

I

Given that $q(\phi,K)$ exceeds unity for some feasible choice of (ϕ,K) and that q is a continuous function, the region over which (i) q exceeds unity, (ii) marginal q declines with increasing K, and (iii) ϕ is positive but less than or equal to unity is a compact set. Within this set, the function K^o assigns to each value of ϕ that value of K for which marginal q equals unity. The projection of this region onto the ϕ axis is the relevant domain for K^o. The only points where K^o may intersect the contour $q=1$ are those where both υ and q equal unity, implying that $D_K q$ is zero (see (4)) and that the tangent to the contour of q is perpendicular to the ϕ axis at these points. Elsewhere the graph of K^o remains strictly inside the contour $q=1$.

From the total differential of (4), assuming υ is constant at unity, the slope of K^o is

$$(A1)\quad D_\varphi K^o = -\frac{D_\varphi \upsilon}{D_K \upsilon} = -\frac{K\, D_\varphi D_K q + D_\varphi q}{2\, D_K q + K\, D_K^2 q}\ .$$

If the slope of K^o is zero, then (from the continuity of q)

$$(A2)\quad K\, D_K(D_\varphi q) = -(D_\varphi q)\ .$$

(A2) would be satisfied if $D_\phi q$ took the functional form $f(\phi)/K$. But this essentially requires that Q, pdf_s, and pdf_c be independent of K. In general, satisfying (A2) over an open interval of ϕ requires that q be independent of ϕ -- τ and δ are zero.

Although K^o generally is not horizontal over an open interval in the domain of ϕ under the conditions of this paper, $D_\phi K^o$ may be zero at one or possibly more points. The optimum choice of (ϕ,K^o) may correspond to one of these points only under special circumstances. If K^o is tangent to a contour of q at a point where both are horizontal to the ϕ axis (from (18) and (19), interchanging the order of differentiation in the numerator of (19)):

$$(A3)\quad D_\varphi q = -(\tau D_\varphi \mu + D_\varphi \lambda) = 0$$

$$(A4)\quad D_K D_\varphi q = -D_K\,(\tau D_\varphi \mu + D_\varphi \lambda) = 0.$$

$\tau D_\phi \mu$ and $D_\phi \lambda$ must be equal in magnitude but of opposite signs, and these slopes must change at the same rate (in opposite directions) when K changes. These two conditions generally will not be satisfied at the same point for independent specifications of the functions *pdf*, as is illustrated by the example in section III.

Because K^o is not necessarily horizontal at the optimal choice of ϕ: <u>given K</u>, $D_\phi q$ does not necessarily equal zero, and the marginal tax saving associated with a small alteration of leverage does not equal the marginal change in the cost of debt financing. Therefore the optimal choice of (ϕ,K) ordinarily is not attained by: first, maximizing q with respect to ϕ given K, thereby defining $\phi^o(K)$; second, choosing K so that $v(\phi^o,K)$ equals unity. With this approach, ϕ^o intersects the contours of q only where their tangents are parallel to the ϕ axis. Therefore, this alternative cannot yield the optimal choice of (ϕ,K), unless K^o happens to be horizontal at the optimal choice of ϕ.

II

q may depend on the enterprise's choice of labor as well as capital. Suppose

$$(A5)\quad r(L,K) = \frac{P(Q(L,K))\;Q(L,K) - W(L)\;L}{K} = \frac{REV(L,K)}{K}$$

$$(A6)\quad q(L,K) = \frac{r(L,K)}{\rho}$$

$$(A7)\quad \upsilon(L,K) = q(L,K) + K\;D_K q(L,K) = \frac{D_K REV(L,K)}{\rho}\ .$$

For simplicity, ρ is a constant, and the price of capital goods is \$1. If the downward-sloping demand curve P, the production function Q, and the upward-sloping supply curve W have the usual properties, and if q exceeds unity for some choice of (L,K), then the region over which q exceeds unity is a compact, possibly convex set.

The optimal choice of K, given L, equates marginal q with unity, $K^o(L)$. Because q is not independent of L, K^o is not horizontal over any open interval of L, unless (following the logic of (A1) and (A2))

$$(A8)\quad D_L q = \frac{D_L REV}{K\;\rho} = \frac{f(L)}{K}\ ,$$

which essentially requires that Q be independent of K. Analogous to (18) and (19), the slopes of contours of q and K^o are, respectively,

$$(A9)\quad D_L K|_{q=const} = -\frac{D_L q}{D_K q} = -\frac{D_L REV}{D_K REV - r}$$

$$(A10)\quad D_L K^o = -\frac{D_L \upsilon}{D_K \upsilon} = -\frac{D_L D_K REV}{D_K^2 REV}\ .$$

At the optimal choice of K, both denominators are negative (the marginal return on capital falls with increasing K, and it is less than the average return on capital). Therefore, both numerators are positive in order for these slopes to be equal at the optimal choice of K.

A change in conditions that diminishes the average return on capital more than the marginal return, thereby making *REV* less concave, tends to increase both of these slopes, because both denominators become less negative. In this case, as in the example in section III, marginal q might fall while the optimal stock of capital rises. Consequently, the analysis of the choice of the optimal (L,K) and the correspondence between changes in marginal q and the optimal stock of capital for (L,K) are similar to those for (ϕ,K).

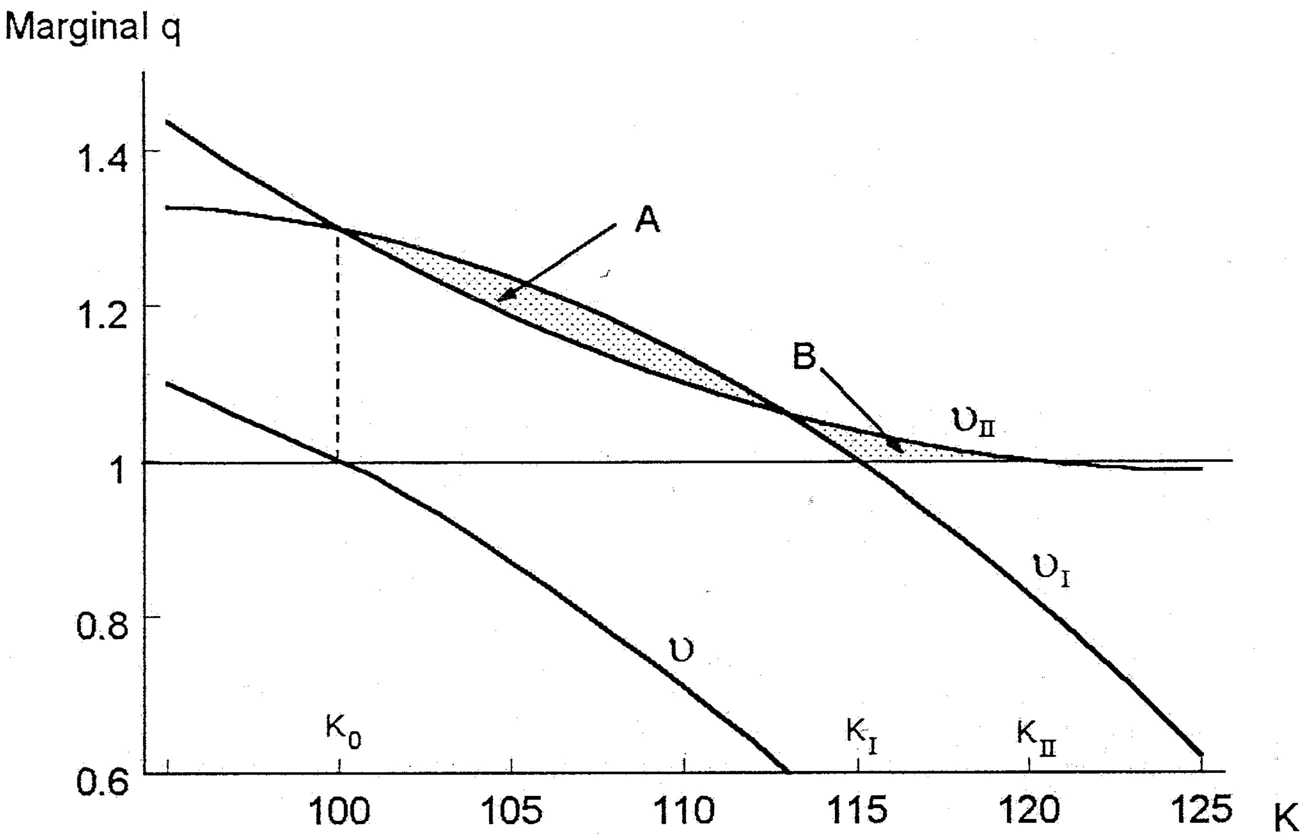
Table 1

The Demand for Capital and the Optimal Choice of Financial Structure

	Initial *REV* Evaluated at Point 1	Revised *REV* Evaluated at Point 1	Revised *REV* Evaluated at Point 2	Revised *REV* Evaluated at Point 3
Stock of Capital (K)	1019.8	1019.8	1474.1	600.0
Equity Financing (ϕ)	50%	50%	86%	85%
Tobin's q				
Average (q)	2.48	1.00	1.01	1.02
Marginal (v)	1.00	.99	1.00	1.01
	in percent			
Marginal Return on Capital (REV')	21.4	29.9	29.1	30.9
Marginal Cost of Capital	21.4	30.9	29.1	29.9
Interest Rate on Debt				
Before Taxes (i)	6.3	18.2	5.9	6.1
After Taxes ($(1-\tau)i$)	3.2	9.1	3.0	3.1
Interest Premium Paid by Shareholders (δ)	0.6	10.8	0.9	1.1
$D_\phi q \ (= -(D_\phi\lambda + D_\phi\tau\mu))$	2.2	3.4	0.9	1.1
$D_\phi\lambda$	-8.6	-15.7	-3.7	-4.1
$D_\phi\tau\mu$	6.3	12.1	2.7	2.9

The Initial REV case is described in (20), (21), and (22). The Revised REV case is described in (21), (22), and (23). The column headings refer to points in Figures 4 and 5. For columns 1 and 3, the discussion preceding proposition 2 describes the optimal strategy; for column 4, the discussion before (24) describes the optimal strategy, given that equity financing cannot exceed \$510. The marginal cost of capital is defined by (16).

Figure 1



$$\upsilon_I(K) = (-.4914 + .0134\,K - 7.143\times10^{-5}\,K^2)\;/\;.1$$

$$I: \quad q^{obs}(100) - q(100) = \int_{100}^{115} (\upsilon_I(K) - 1)dK\;/\;100 = 2.7\%$$

$$\upsilon_{II}(K) = (.880 - .0125\,K + 5.00\times10^{-5}\,K^2)\;/\;.1$$

$$II: \quad q^{obs}(100) - q(100) = \int_{100}^{120} (\upsilon_{II}(K) - 1)dK\;/\;100 = 2.3\%$$

Figure 2

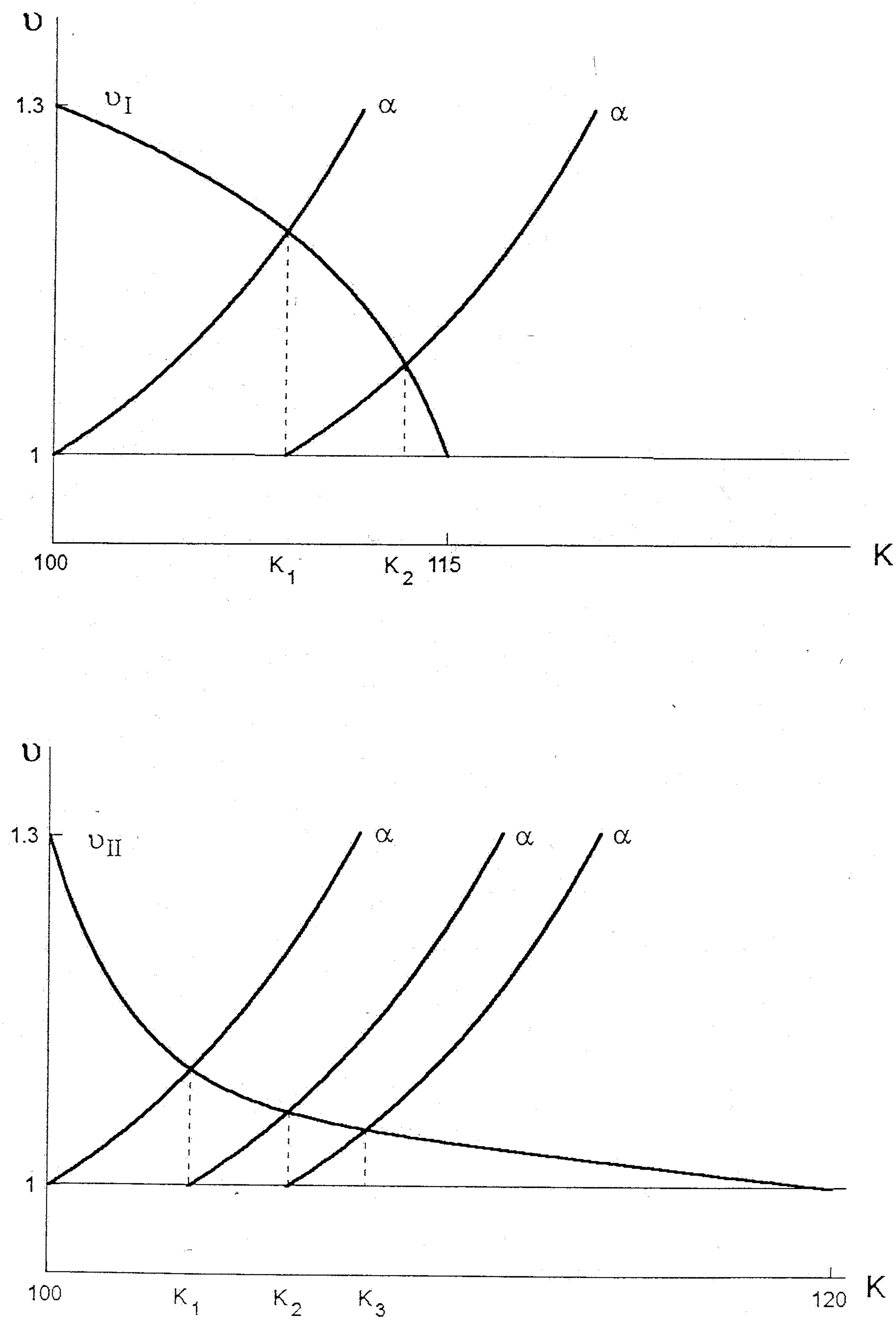


Figure 3

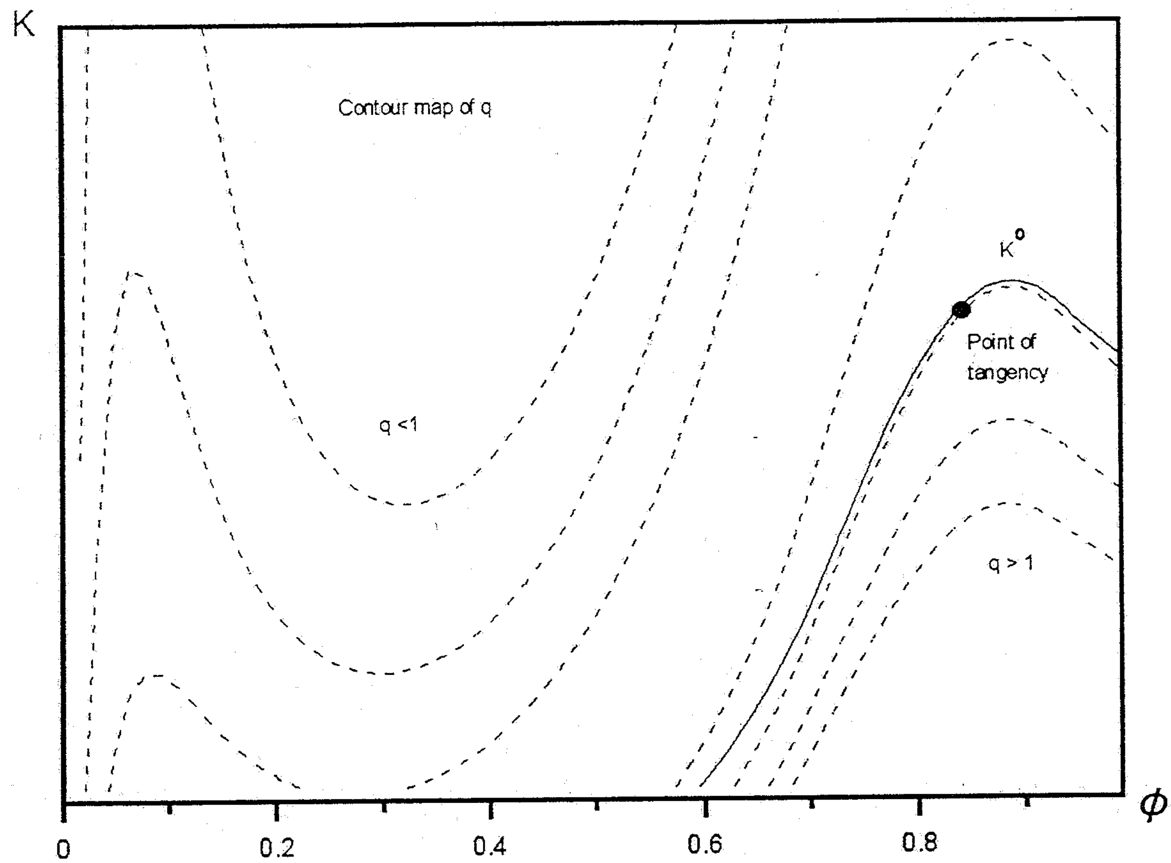
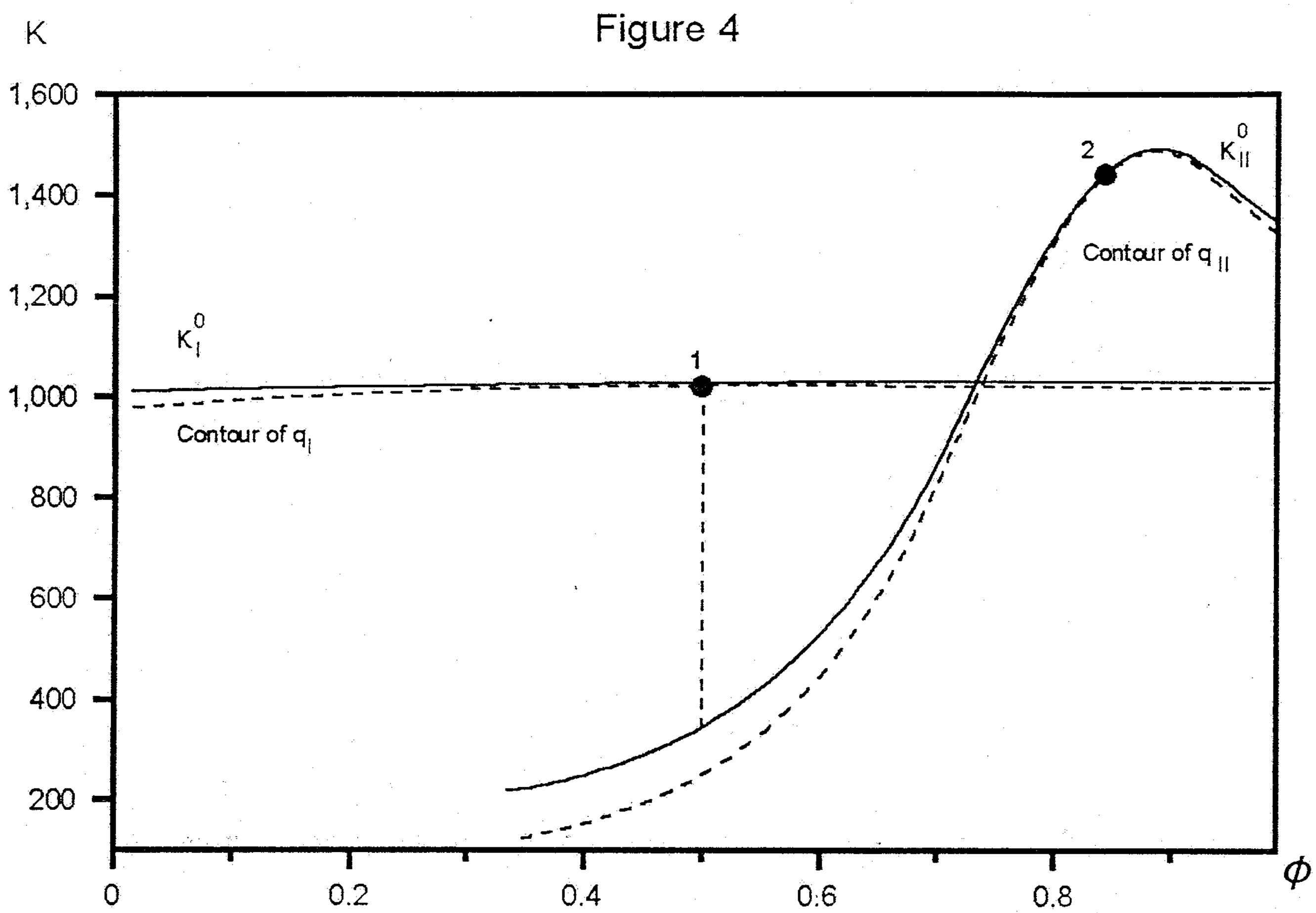


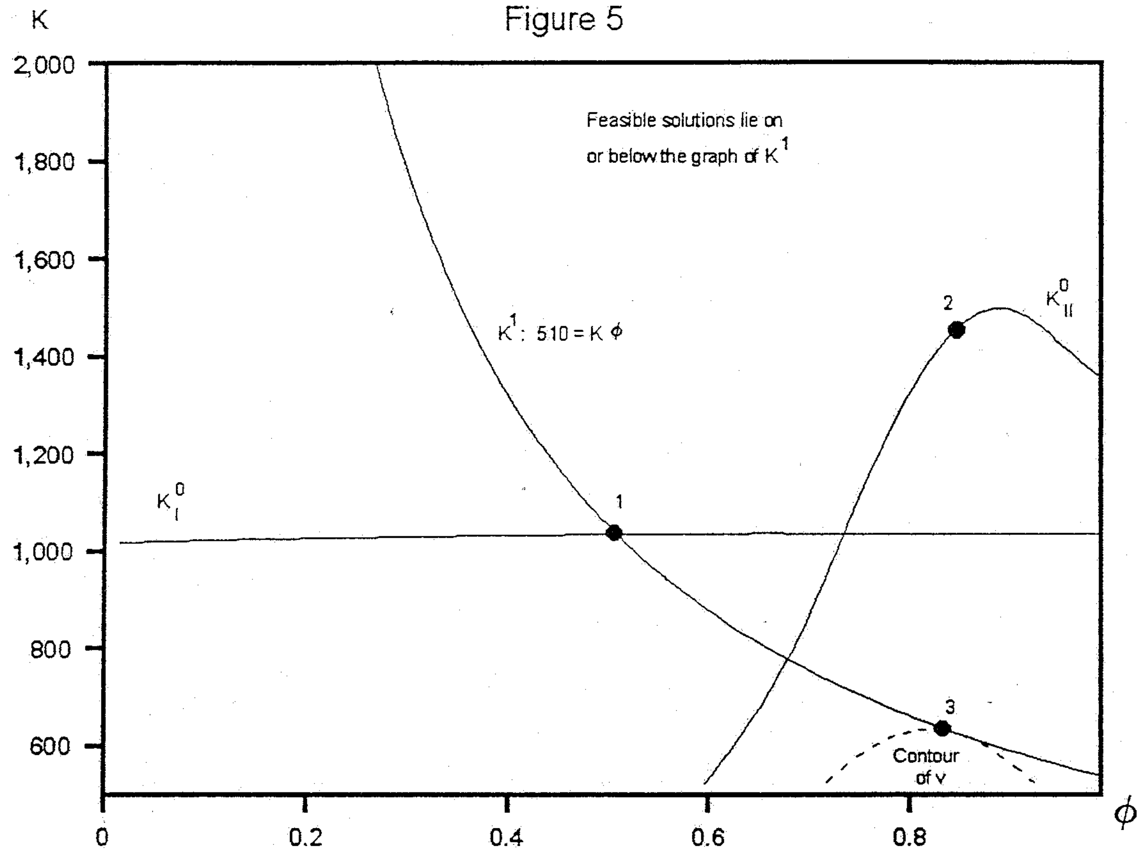
Figure 4



I: Specified in expressions (22), (23), (24), (8), (14), and (16). See Table 1 column 1.

II: Specified in expressions (23), (24), (25), (8), (14) and (16). See Table 1 column 3.

Figure 5



See Figure 4 and Table 1.

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