Are "Deep" Parameters Stable?
The Lucas Critique as an Empirical Hypothesis

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Abstract

For years, the problems associated with the Lucas critique have loomed over empirical macroeconomics. Since the publication of the classic Lucas (1976) critique, researchers have endeavored to specify models that capture the underlying dynamic decision-making behavior of consumers and firms who require forecasts of future events. By uncovering the "deep" structural parameters that characterize these fundamental behaviors, and by explicitly modeling expectations, it is argued, one can capture the dependence of agents' behavior on the functions describing policy. However, relatively little effort has been devoted to testing the empirical importance of this critique. Can one find specifications that are policy-invariant? This paper develops a set of tests for small macroeconometric models, especially those used for monetary policy analysis, and implements them on a set of models used extensively in the literature. In particular, we attempt to test the robustness of optimizing versus non-optimizing models to changes in the monetary policy regime. In this paper we present evidence that shows that some forward-looking models from the recent literature may be less stable than their better-fitting backward-looking counterparts.

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1. Introduction

The Theory of the Lucas Critique

In his seminal 1976 paper, Lucas discusses the problem of econometric forecasting in an economy in which the behavior of policymakers may shift across time.[1] When private agents are forward looking, their decisions will depend in part upon their forecasts of the future actions of policymakers. When the relationship describing systematic policy actions changes in a way that is observable by private agents, their forecasts of future policy actions should change conformably. Well-specified econometric models should reflect this linkage, Lucas argues. If they do not, then the models' forecasts may themselves exhibit instability across time when shifts in policy regime occur.

In Lucas's notation, the forecasting model may be represented by a function *F* that links forcing variables x_t, including policy variables, with endogenous variables y_t, subject to random shocks ε_t, with the parameters of *F* collected in the vector **1**:

$$y_{t+1} = F(y_t, x_t, \Theta, e_t) \tag{1.1}$$

A policy action entails a choice of at least some components of x_t. If forecasts from the model are to be useful, the form of *F* and its parameters **1** must be invariant to changes in the process generating x_t, which for our purposes means that *F* and **1** must be invariant to changes in the systematic component of monetary policy.

Lucas's critique may be summarized in his assertion that, for the backward-looking models that were conventional at the time, "Everything we know about dynamic economic theory indicates that this presumption [that *F* is stable across policy shifts] is unjustified" (Lucas 1976, p. 111). If the *F* in backward-looking models is to be stable across regime changes, then agents' views about future movements in *x* cannot change even when the underlying process generating *x* changes. Lucas views this as unlikely. More likely in Lucas's view is that **1** will depend on the parameters that govern the behavior of policy.

Thus, suppose policy makers choose elements of x_t according to a rule

[1] See Lucas (1976). Of course, the critique applies to shifts in the behavior of *any* of the agents in the model.

$x_t = G(y_t, \lambda, \eta_t)$, where policy parameters are denoted 8 and η_t is a vector of disturbances. Then the goal of economic modelers should be to estimate functions that make the linkage between 8 and 1 explicit, that is they must find stable representations $F(y_t, x_t, 1(8), ,_t)$.

Lucas's critique was directed at the users and purveyors of econometric models at the time, many of which relied on simple backward-looking descriptions of expectations formation which would *not* change even when monetary policy changed. Suppose, for example, that agents in an econometric model form expectations about inflation B_t using a simple autoregression

$$p_t^e = \sum_{i=1}^{k} a_i p_{t-i} \tag{1.2}$$

based on the behavior of inflation under monetary regime I. The coefficients $"_i$ would approximate the time series behavior of B_t under regime I. When the systematic behavior of monetary policy changes in new regime II, in general the coefficients $"_i$ that describe the approximate time series behavior of B_t will also change. Models that assume that agents will continue to form expectations using the $"_i$ estimates from equation (1.2) will be missing a potentially important link between 1 (here the $"_i$) and 8 (the parameters that capture the shift in policy behavior). Lucas suggests that econometric modelers need to be wary of such backward-looking specifications, as they may not be stable across changes in policy regimes.

Of course, the model that incorporates equation (1.2) as its inflation expectations specification *could* be stable across observed shifts in monetary regime. Stability could arise for two reasons: (1) the observed shifts in monetary policy have been relatively modest, or have resulted in an autoregressive representation for inflation that differs insignificantly across regimes; equivalently, the link between 1 and 8 is not empirically important in the historical sample; or (2) agents actually form their expectations according to the estimated version of equation (1.2), and do not vary that expectations mechanism across monetary regime shifts, and thus the model will be stable.

In this regard, a key point that this paper wishes to re-emphasize[2] is that for any particular specification, ***the Lucas critique is a testable empirical hypothesis.*** We cannot know *a priori* whether observed shifts in policy have been large enough to alter significantly the backward-looking representations of economic variables. Similarly, we cannot know *a priori* how agents form their expectations of future events. As a result, the stability or instability of backward-looking models is an empirical, not a theoretical issue.

Lucas's Solution to the Problem

Lucas and a host of researchers since the publication of his paper have endeavored to specify models that capture the underlying dynamic decision-making behavior of consumers and firms who require forecasts of future events. By uncovering the "deep" structural parameters that characterize these fundamental behaviors, and by explicitly modeling expectations, it is argued, one can capture the (presumed) dependence of agents' behavior on the functions describing policy.

But just as the backward-looking models cannot be known to be subject to the Lucas critique *a priori*, neither can the "Lucas" solution to the problem be known to be correct *a priori.* A model that is based on the underlying optimizing behavior of firms and consumers and incorporates rational expectations may also be unstable across policy regimes for two reasons: (1) the model may inaccurately reflect the objective function or constraints facing firms and consumers; and (2) the model may inaccurately reflect the way in which agents form expectations. In either case, shifts in policy can cause significant shifts in the model's parameters. Therefore, optimizing, explicit expectations models should also be subjected to empirical tests of cross-regime stability.

In the end, then, the Lucas critique is an empirical issue: Can one find specifications that are policy-invariant? The hope of policy modelers is that we can, but we cannot know in advance whether those specifications will be backward- or forward-looking, rational or near-rational or irrational, or based on optimizing behavior of the textbook variety. Furthermore, as Ericsson and Irons (1995) have documented, it is

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[2] See, e.g., the earlier work of Ericsson et al. (1998), Oliner et al. (1996), Engle and Hendry (1993), Favero and Hendry (1992), Alogoskoufis and Smith (1991), and Miller and Roberds (1991), who have looked at various specific aspects of this issue. Ericsson and Irons (1995) present a thorough survey of papers that cite the Lucas critique.

difficult to find any careful formal testing of the practical significance of the Lucas critique for any of these types of models. In this paper, we present evidence that shows that some forward-looking models from the recent literature may be less stable – more susceptible to the Lucas critique – than their better-fitting backward-looking counterparts.

While we suggest that addressing the Lucas critique will always be an empirical matter, it should be noted that the answers obtained in this way can never be completely satisfying. First, a model that is found to be stable across policy regimes has really only shown its approximate stability across the historical data. The failure to reject stability across observed shifts does not insure stability in the presence of shifts that have not yet occurred. Second, of course, the nature of our statistical tests is such that we cannot prove stability, we can only fail to reject stability. These unavoidable limitations of available methodology make the analysis of alternative policy regimes, a key goal of this line of research, unavoidably hazardous.

2. Policy Reaction Functions and Policy Regimes

Given the principal purpose of this paper – the testing of certain macroeconomic models for stability across changes in monetary policy regime – it is practically essential to give some thought to the construction of a model of monetary policy. Such a model should capture both the systematic operation of monetary policy within one regime and more fundamental changes that occur in the transition from one regime to another. With this in mind, we define in this section a policy reaction function that embodies our assumptions about regime changes and that is later used in section 4 to close the structural models examined there.

The policy reaction function has a general form similar to that of the Taylor (1993) rule in that the policy instrument, the federal funds rate, reacts to the level of inflation and to the output gap. In contrast to Taylor (1993), however, we allow somewhat greater flexibility with regard to functional form and we estimate the function's parameters. Specifically, the equation is

$$\Delta r_t = a(\bar{p}_t - b r_{t-1}) + c y_t + d \Delta r_{t-1} + e_t \,, \qquad (2.1)$$

where r is the quarterly-average federal funds rate, $\bar{p}$ is the level of inflation over the last four quarters as measured by the chain-weighted GDP deflator, y is the real output gap as defined by the Congressional Budget Office, and ε is a random disturbance. This equation may also be motivated, as suggested by Clarida et al. (1999), by viewing it as an error-correction model for the federal funds rate in which the target interest rate is a function of inflation and the output gap.

The first term of equation (2.1) corresponds to the gap between current inflation and the inflation target in Taylor's formulation. Of course, inflation targets are not explicitly available, so we introduce a term containing the lagged federal funds rate to serve as a reference point for inflation. The second term of equation (2.1), as in the Taylor equation, is the output gap. Finally, we include a term containing the lagged first difference of the federal funds rate. Recent research suggests that there is considerable persistence in the process for the federal funds rate.[3] This persistence is modeled in (2.1) by the use of the first difference of the federal funds rate and by the inclusion of the lagged difference as well as the level of the rate in the right-hand side. Full sample estimates of the parameters of equation (2.1) are given in table 1.[4]

Table 1. Estimates of equation (2.1) from 1966:1 to 1997:4 (Quarterly Data)

			With d=0	
Parameter	Value	Standard Error	Value	Standard Error
a	.15	.076	.14	.080
b	.57	.076	.55	.073
c	.15	.043	.17	.042
d	.09	.100	—	—

Note in the table that the value of d is small and not statistically significant. Nevertheless, we estimate the equation and test for breaks both with and without this parameter, since there is some evidence that the parameter may be non-zero in some subsamples.

The identification of break dates in this equation is clearly important, especially in view of the focus of this paper. We look for evidence of such changes in the estimates of equation (2.1) by performing various tests of stability. In addition, we have information

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[3] See, e.g., Rotemberg and Woodford (1998).

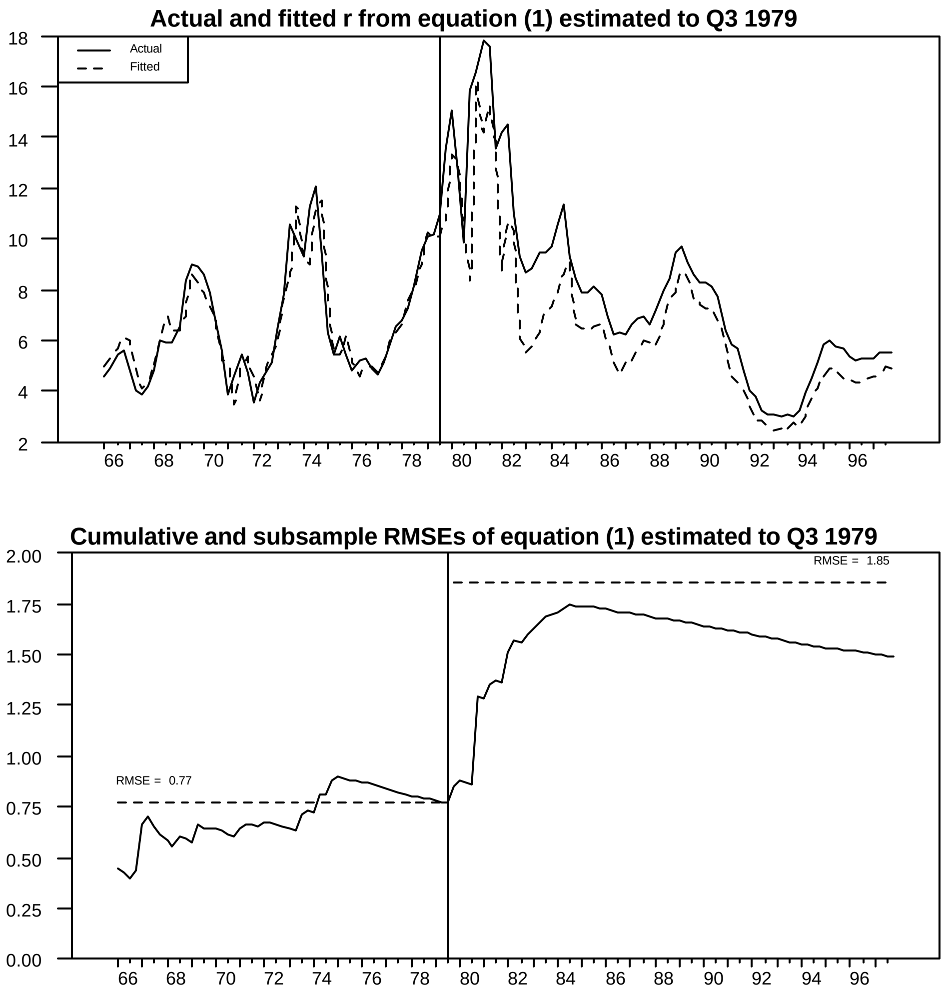
[4] Since there is evidence of autocorrelation in the residual, consistent standard errors are calculated using the Newey-West (1987) method with 4 lags. Otherwise, the estimates are obtained by least squares on the assumption that the contemporaneous endogenous variables on the right-hand side are not affected within

concerning actual changes in monetary policy and in operating procedures, and we make use of those priors in the testing of equation (2.1). Specifically, we look for breaks after 1979:3, 1982:3, and 1987:2. The first and second dates are associated with changes in Federal Reserve operating procedures. October 1979 is of course also well known as the start of the tenure of Chairman Volcker, while 1982 marked the return to operating procedures similar to those of the pre-1979 period. The third date corresponds to the appointment of Chairman Greenspan, and there is at least some casual evidence that changes at that time may have noticeable effects on the data.

Before proceeding to the formal tests of stability, it is instructive to perform a simple experiment with equation (2.1). Suppose the equation is estimated with data up to 1979:3 and the parameters then used to construct fitted values of the interest rate for the remainder of the estimation period. What is the magnitude of the errors that result? This simple experiment produces straightforward evidence that the equation underwent some sort of structural shift in 1979. The top panel of figure 1 shows the actual and fitted values from this experiment, and it is easy to identify a change visually. The bottom panel shows cumulative root-mean-square errors (from 1966:1 to the given quarter) and subsample root-mean-square errors. Note that the RMSE in the second subsample is about twice that in the first, which reinforces the conclusions drawn from the first panel.

the same quarter by the dependent variable, the current interest rate. Experiments with instrumental variables estimation yielded almost identical results.

Figure 1



Formal tests of stability produce similar results. We employ in this section two test statistics, a Lagrange multiplier test proposed by Andrews and Fair (1988) and a predictive test proposed by Ghysels and Hall (1990). Both of these tests seem to have reasonable characteristics in sample sizes like ours.[5] The classical LM test and its properties are well known in the literature. We estimate under the null of no break and look at the departure from zero of the orthogonality conditions in the two subsamples.

[5] In simulations we performed with stylized models with autoregressive properties similar to our actual models, the commonly used Wald test (which roughly corresponds to a Chow test) had a size much larger than the nominal size. E.g., the true size of a 5% test with 100 observations was 17%. Given the somewhat inconsistent results provided by break tests, it seems useful to show results for both the LM and TS tests.

The TS statistic of Ghysels and Hall (1990) has the same asymptotic distribution as the LM statistic and has the interpretive advantage of corresponding to our simple predictive experiment above. Specifically, the model is estimated in the first subsample, up to the possible break point, and departures from zero of the orthogonality conditions in the second subsample are tested. The specific forms of these statistics are given in appendix 1 and the results are presented in table 2. The table shows results both with and without the lagged change in the interest rate in the right-hand side of equation (2.1).

Table 2. Stability tests of equation (2.1) from 1966:1 to 1997:4 (*p* values)

			With d=0	
Break point	LM	TS	LM	TS
1979:3	.263	.078	.198	.046
1982:3	.376	.303	.425	.303
1987:2	.945	.363	.989	.256
Unknown	.317	.241	.547	.213
Max	1980:3	1980:3	1980:3	1975:2

In line with the earlier heuristic results, the TS statistic provides evidence of a break at 1979:3 at the 5% level when d=0 and at the 10% level when d is unconstrained. Although this evidence is not supported by the LM statistic, it seems reasonable to assume that a change did take place in light of events at the Federal Reserve in October 1979. For the other two possible break dates, the tests fail to reject. The tests for an unknown break date also fail to reject. Note that, for technical reasons, these tests require excluding some fraction of the observations at the beginning and end of the sample, and we exclude 25% at each end.[6] The date identified by the LM test is very close to one of our “a priori” break dates, but the evidence of a break provided by these tests is fairly weak.

What do we conclude from the foregoing tests? First, it seems fairly clear that we must allow for a break around 1979:3. In that case, we know important changes took place and the statistical evidence is significant. For the other two break dates, we take a more agnostic approach. On the one hand, we know real changes took place, on the other hand, there seems to be little statistical evidence that our equation was affected by those changes. In the tests of the macroeconomic equations that follow in subsequent sections,

[6] See explanation in Andrews (1993).

we err on the side of assuming that there are breaks at those points. If the changes are not there, consistent parameter estimates should lead to similar values before and after the break date.

3. Empirical Analysis of Single-Equation Models

We examine in this section the stability of two basic equations that are commonly included in macroeconomic models: a Phillips curve and an IS curve. The key comparison for each of these equations is that between a forward-looking version of the equation, which includes an expected future value of the dependent variable in the right-hand side, and a backward-looking version that only contains lagged values in the right-hand side. Previous research (e.g., Estrella and Fuhrer (1998)) has shown that the backward-looking equations tend to fit the data better. However, are they more susceptible to structural breaks than the forward-looking equations, as theoretical arguments in Lucas (1976) suggest?

We start by defining the forward-looking equations. Because we want to test models that have been seriously considered in the literature, and not straw men, we select our specifications by matching them as closely as possible to existing models. In the case of the single-equation Phillips curve, the specification is adopted from Roberts (1995), who derives the form of the equation from models with optimizing agents. The equation is

$$p_t = E_t p_{t+1} + b_0 + b_1 y_t + e_t . \qquad (3.1)$$

Since the right-hand-side variables in (3.1) are endogenous and expectations are rational, Roberts estimates this equation by substituting actual future inflation for expected inflation, and then using as instruments the log-change in oil prices, its first lag, the log-change in real federal government purchases, and a dummy variable indicating whether the President is a Democrat. We also adopt this estimation strategy, and we measure inflation by the growth in the CPI as in Roberts (1995).

The form of the forward-looking IS curve is obtained from McCallum and Nelson (1998), who also derive it by optimization. The IS curve is

$$y_t = E_t y_{t+1} + c_0 + c_1 (r_t - E_t p_{t+1}) + e_t . \qquad (3.2)$$

For consistency, we apply the same single-equation estimation method and the same data used by Roberts (1995) for the Phillips curve, and find that the same instruments seem reasonable in this case. Full-sample results for the PC and IS equations are presented in table 3.

Table 3. Estimates of forward-looking equations (3.1) and (3.2) from 1966:1 to 1997:4

	Equation (3.1): Phillips curve	
Parameter	Value	Standard Error
b_0	.060	.144
b_1	.082	.186
	Equation (3.2): IS curve	
c_0	.15	.16
c_1	-.056	.061

The results in table 3 are disappointing. Although the parameter estimates have the “right” signs, the magnitudes and the levels of significance are very low, suggesting that the models are unsatisfactory. These results are consistent with earlier estimates in Roberts (1995) and Estrella and Fuhrer (1998). McCallum and Nelson (1998) obtained somewhat stronger significance in their estimates of the IS equation.

As noted, earlier work has shown that backward-looking equations fit the data better. Consider the following backward-looking PC and IS equations, which are modeled on the ones estimated by Rudebusch and Svensson (1998). These equations are essentially a subset of a constrained vector autoregression (VAR). The PC is

$$\pi_t = b_0 + b_1\pi_{t-1} + b_2\pi_{t-2} + b_3\pi_{t-3} + b_4\pi_{t-4} + b_5 y_{t-1} + \varepsilon_t \qquad (3.3)$$

and the IS curve is

$$y_t = c_0 + c_1 y_{t-1} + c_2 y_{t-2} + c_3(\bar{r}_{t-1} - \bar{\pi}_{t-1}) + e_t\,, \qquad (3.4)$$

where both the federal funds rate and inflation enter in four-quarter averages. Rudebusch and Svensson define inflation in terms of the chain-weighted GDP deflator, rather than the CPI, as in the forward equations, and we adopt the same definition for our single equation estimates and formal tests.[7] Estimates of equations (3.3) and (3.4) are shown in table 4, with Newey-West (1987) standard errors using 4 lags.

Table 4. Estimates of backward-looking equations (3.3) and (3.4) from 1966:1 to 1997:4

	Equation (3.3): Phillips curve	
Parameter	Value	Standard Error
b_0	.148	.202
b_1	.702	.109
b_2	-.109	.117
b_3	.288	.092
b_4	.100	.088
b_5	.144	.037
	Equation (3.4): IS curve	
c_0	.138	.112
c_1	1.18	.106
c_2	-.283	.104
c_3	-.078	.038

The parameters in table 4 are clearly more tightly estimated than those of the forward-looking models in table 3. By practically any measure, the fit of the backward-looking equations is superior to that of their forward-looking counterparts. For instance, the R^2s of equations (3.1) and (3.2) are -0.05 and -0.08, respectively, as compared with 0.81 and 0.90 for equations (3.3) and (3.4). This measure is not entirely appropriate with the instrumental variable estimates of the forward-looking equations, but the large differences are indicative of the true fit of the equations.

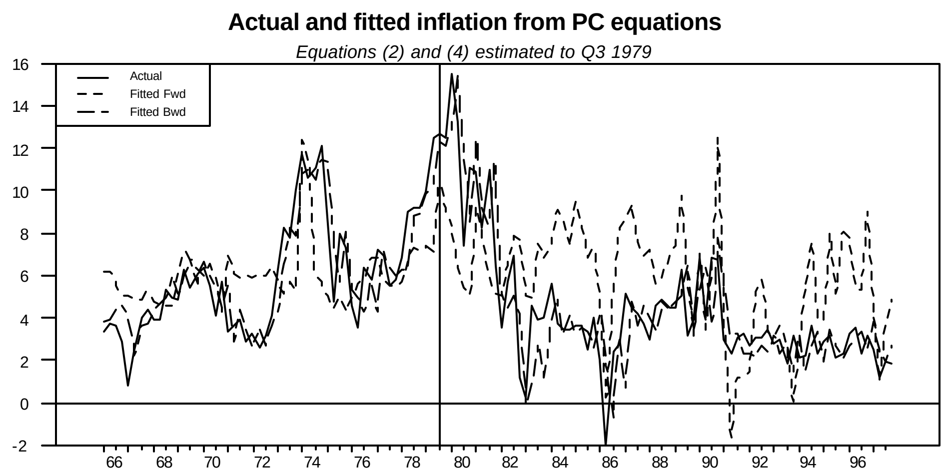
However, it is possible that notwithstanding their inferior fit, the forward-looking equations may be more stable across changes in monetary policy regime. There are at least two reasons for this. First, the forward-looking equations were developed in an optimizing rational expectations framework, precisely to attempt to deal with the problem identified by the Lucas (1976) critique. Second, a more mechanical reason is that when the parameters are estimated much less tightly, it may be more difficult to detect a break

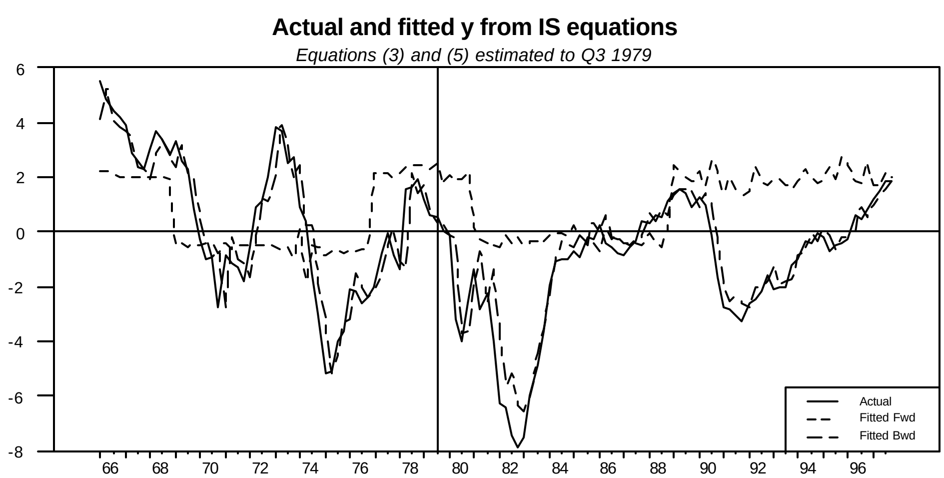
[7] For some purposes, it will be useful to have the same definition of inflation in the forward and backward models. Thus, we settle on CPI inflation in figure 2 and on the GDP deflator in the multi-equation model of section 4.

given the large margin for error embedded in the estimates. Thus, we use the techniques described in the previous section and in appendix 1 to test for structural breaks in these equations that may have resulted from structural changes in monetary policy.

To start, it is helpful to examine simple visual evidence of possible breaks, and we do this by performing the same type of predictive experiment that was applied in the previous section to the policy reaction function. Thus, each of equations (3.1) to (3.4) is estimated with data from 1966:1 to 1979:3 and the actual and fitted values are compared over the full sample period (to 1997:4). The results are shown in figure 2.

Figure 2





Some of the features of figure 2 are visually clear, for instance, the fact that the fit of the forward equations is poorer than the fit of the backward equations. Given that large difference, however, it is less obvious whether the fit in the second “out-of-sample” subperiod represents a deterioration as compared with the fit in the first “in-sample” subperiod. If we take a look at the formal statistical evidence, we see that the signals are not very strong, but that the clearest evidence of such deterioration is for the forward-looking PC.

Table 5 contains the evidence for the PC equations.[8] The only rejection of stability at standard significance levels is found in the case of the 1979:3 break point, for which the LM statistic is significant at the 10% level. The test for an unknown break point comes very close to matching the date of this break, although it is not significant at the 10% level. Even though the evidence is not strong, it is somewhat surprising to find any such evidence at all, given the relatively large standard errors of the estimates of the parameters of the forward looking PC equation, which were reported in table 2.

Table 5. Stability tests of PC equations (3.1) and (3.3) from 1966:1 to 1997:4 (*p* values)

	Forward-looking (3.1)		Backward-looking (3.3)	
Break point	LM	TS	LM	TS
1979:3	.073	.113	.543	.632
1982:3	.238	.133	.911	.616
1987:2	.389	.140	.711	.686
Unknown	.122	.332	.962	.956
Max	1979:4	1977:2	1988:2	1981:2

The corresponding results for the IS equation are reported in table 6. In this case, there are no significant rejections of stability, and perhaps the most interesting feature of this table is that the dates in which the statistics assume their maximum values are all within the period in which the Federal Reserve operating procedures were changed.

[8] Note that in applying the tests for unknown break points to these equations, we exclude 30% of the observations at each end, rather than 25% as for the reaction function. The reason is that the dummy variable indicating a Democrat as President must not be constant in any of the subsamples.

Table 6. Stability tests of IS equations (3.2) and (3.4) from 1966:1 to 1997:4 (*p* values)

	Forward-looking (3.2)		Backward-looking (3.4)	
Break point	LM	TS	LM	TS
1979:3	.855	.527	.860	.403
1982:3	.492	.193	.146	.196
1987:2	.873	.277	.673	.617
Unknown	.928	.669	.521	.573
Max	1980:2	1982:4	1982:4	1980:3

With the above single-equation tests, it is difficult to make a clear distinction between the performances of the forward- and backward-looking models. If anything, the test evidence suggests that the backward-looking models have a slight edge, but that structural stability is not a major problem for either type of model. In the following section, we make an attempt to sharpen these results by bringing more information to bear on the problem in the context of a structural system estimated based on full-information maximum likelihood.

4. Empirical Analysis of Forward- and Backward-Looking Multi-Equation Models

The preceding section analyzes the stability of forward- and backward-looking models in a single-equation context. However, there are two reasons for examining the stability of the models in a system or multi-equation context. First, the models are designed for evaluating alternative monetary policy regimes via simulations and welfare computations. Doing so relies on the interactions among the monetary policy rule, the IS curve specification, and the price or Phillips curve specification. Thus, we would like to know whether the joint behavior of the model's equations exhibits evidence of instability across shifts in monetary policy regime. Evidence of instability in the overall model would make the model less attractive for evaluating alternative monetary policies, even if individual equations show little sign of instability.

Second, we should expect to improve our ability to detect structural shifts by using full rather than limited information methods. Individual equations' marginally significant instabilities may imply more significant instability of the whole model. It is only possible to detect such system instabilities by working with the full model specification. We pursue this strategy in this section.

We examine three archetypal models, comprising an inflation specification, a monetary policy rule for the short-term interest rate, and an aggregate demand or IS specification. The monetary policy rule is identical for all models, and takes the form described above in equation (2.1). The first model incorporates backward-looking functions for inflation and the IS curve, as in equations (3.3) and (3.4) above. The other two models employ forward-looking inflation and IS specifications, as in equations (3.1) and (3.2). We also examine a variant of the forward-looking model that adds serially correlated shocks to the inflation and IS specifications. The error specification that we use explicitly models the shocks as second-order autoregressive processes, which appears to be sufficient to reduce the residuals to white noise.[9]

We examine two types of stability tests, as in section 3 above. First, we look at the out-of-sample predictions made by the system of equations across potential regime breaks. Second, we examine statistical tests of stability, focusing primarily on the likelihood ratio test. We prefer the likelihood ratio test as it follows naturally from the maximum likelihood estimation procedure that we employ for system estimation.[10]

Graphical Evidence of Instability

Figures 3 to 5 display static simulations of each of the three models. The models are estimated via maximum likelihood over the sample 1966:1-1979:3, and then simulated (single-period or static simulations) within that sample and past the estimation sample through 1997:4. The estimation procedure is described in appendix 2. The simulations are full system simulations, imposing rational expectations (for the forward-looking models).[11]

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[9] To the extent that the dynamics of the model are driven by the autocorrelated error processes, the forward-looking model begins to look more like the backward-looking, loosely constrained vector autoregressive model.

[10] We have experimented with a number of other tests, including the LM, Wald, and Ghysels-Hall (1990) tests, exploring their finite sample properties and robustness to different approximations to the asymptotic covariance matrices required for each. Some information on our experience with these tests is available from the authors.

[11] Note the difference between these simulation paths and the single-equation simulations presented in section 3 above. Here, the cross-equation constraints implied by rational expectations are imposed, producing rather stark dynamic implications, particularly for the simple forward-looking model. In the single-equation simulations presented above, expectations are proxied by unconstrained least-squares projections on instrument sets. This less-constrained expectations framework allows the expectations to conform more closely to the data than the expectations that are formed in a model-consistent fashion.

As figure 3 indicates, the forecast for the simple forward-looking model with *iid* errors varies almost imperceptibly over time. In both inflation and IS specifications, no initial conditions will alter the path of the endogenous variable. Both are expected to remain at their steady-state values. The estimated shocks in the model equal the data. In these circumstances, it is very difficult to detect visually a structural break in the predictions of the model.

Figure 3

Out–sample fit, Forward–Looking Model, iid errors

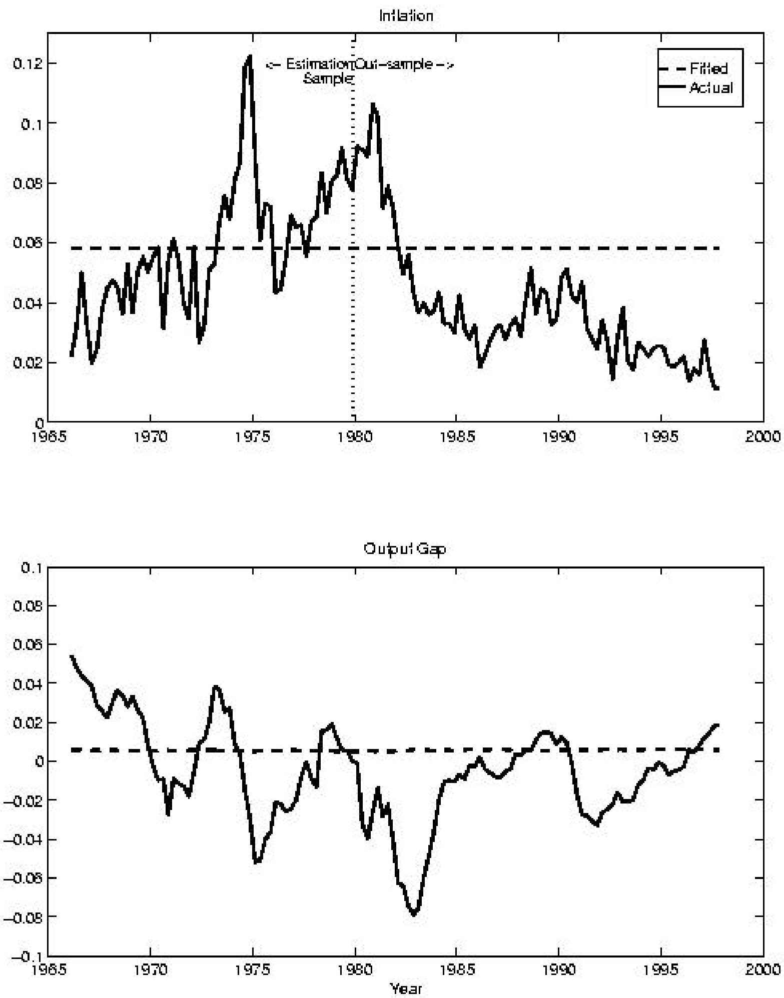
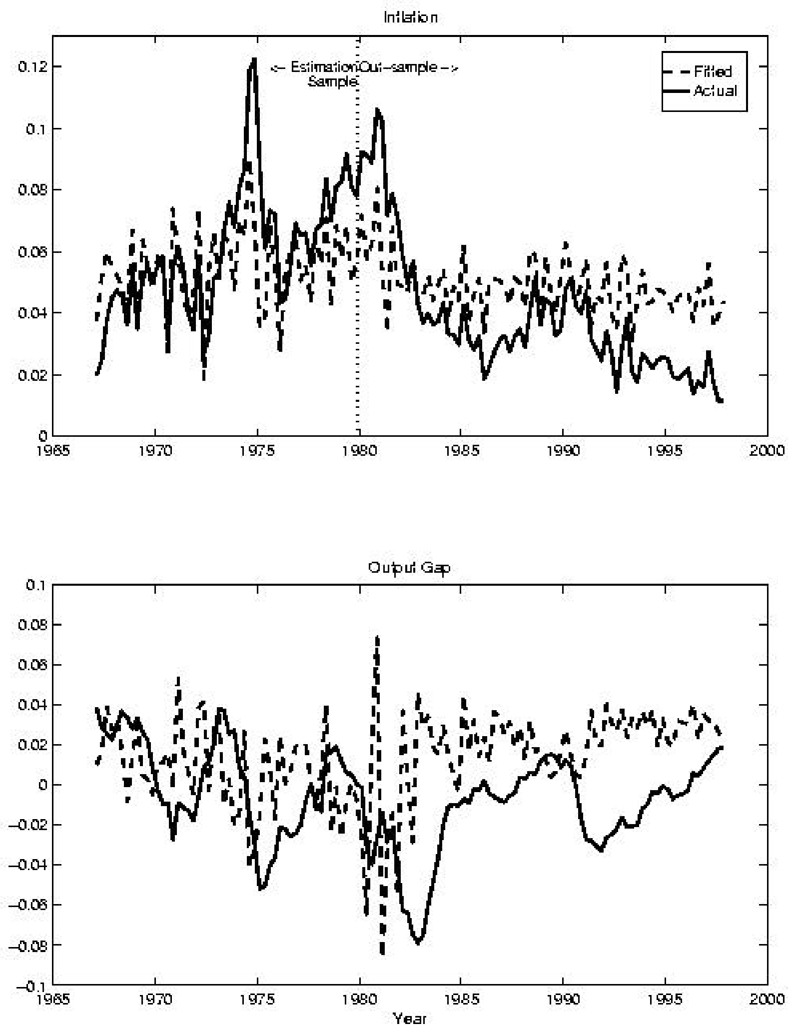


Figure 4 displays the same results for the forward-looking model with AR(2) errors. Now the static simulation exhibits some conformity between model predictions and data. The figure also suggests, however, that the one-period predictions begin to drift away from the actuals around 1979-80. Both inflation and the output gap are systematically overpredicted by the model from 1980 forward.[12]

Figure 4

Out–sample fit, Forward–Looking Model, AR(2) errors

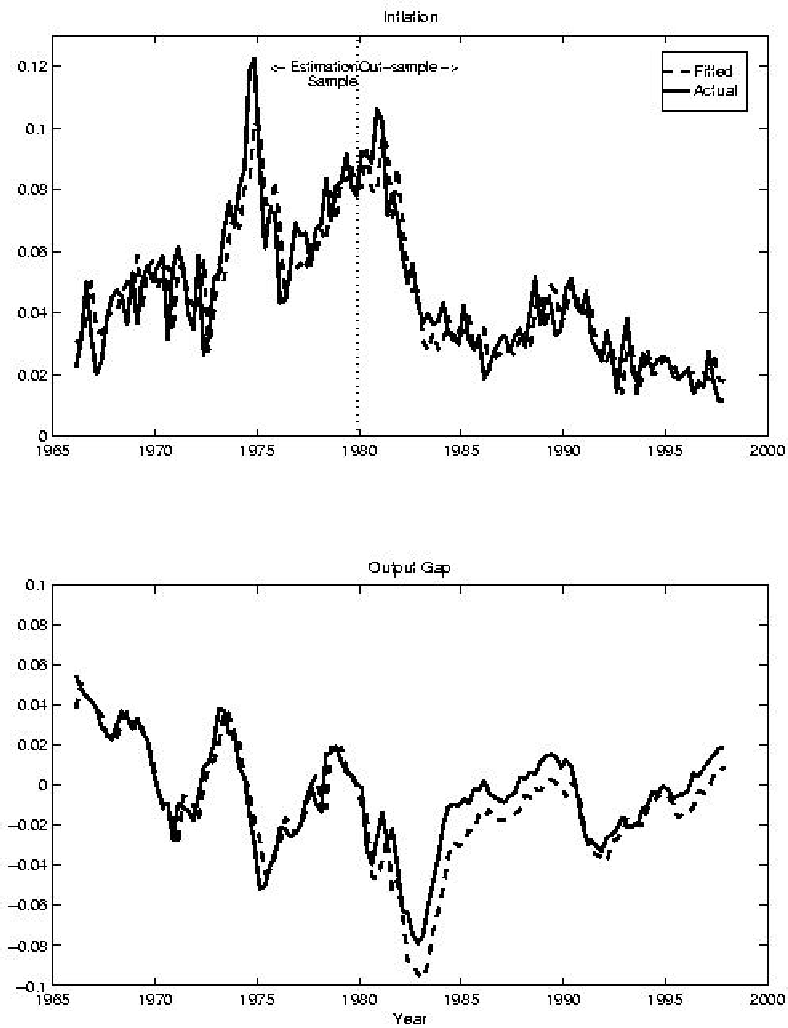


In contrast, figure 5 displays the simulated values from the backward-looking model. As in the preceding figures, the model is estimated from 1966 through 1979. The predicted values for inflation lie quite close to the actuals both before and after 1979. The

[12] Use of the output gap avoids the requirement that agents in the model know about the shift downward in the trend rate of output growth in the mid-1970s. Thus, this important shift cannot account for the failure of the models to predict outcomes out of sample in the 1980s and forward.

predicted values for the output gap remain very highly correlated with the actual after 1979, although there appears to be an intercept shift in the later sample.

Figure 5
Out–sample fit, Backward–Looking Model



The correlations between predicted and actual values for inflation and the output gap for the three models and two samples appear in table 7. In the second sample, the backward-looking model’s predictions for both inflation and output exhibit greater than 90 percent correlation with the actuals. As indicated in the top panel of the table, the correlation between predicted and actual inflation for the forward-looking model with *iid* errors is *negative* in both samples. The simple forward-looking model's predictions for the output gap exhibit correlations of no more than 25 percent with the actuals in both samples. The error-augmented forward-looking model's inflation predictions are much better, exhibiting about 60 percent correlation in both samples, while its output gap correlations are quite poor, below 20 percent in both samples. As noted above, this inability of the forward-looking models to fit the data on inflation and output *within* the estimation sample could make it difficult to detect a breakdown in fit out of sample.

Table 7. Correlations between data and models' predictions of Inflation and Output Gap

Variable	Forward-looking, *iid* errors	Forward-looking, AR(2) errors	Backward-looking
Inflation			
1966:1-1979:3	-.75	.59	.80
1979:4-1997:4	-.90	.64	.93
Output Gap			
1966:1-1979:3	.25	.18	.93
1979:4-1997:4	.18	.11	.94

Likelihood Ratio Tests: Known Breakpoints

In the likelihood ratio tests presented below, we examine sample splits across known breakpoints, determined by both *a priori* knowledge about monetary policy, and by the sample break tests reported in section 2 above. The test procedure that we adopt is as follows. We first estimate the structural model in "unconstrained" form, allowing the parameters in the IS and inflation specification to differ across breakpoints. We employ the full information maximum likelihood estimation algorithm that is detailed in appendix 2. We then constrain the IS and inflation equation parameters—but not the reaction function parameters—to be equal across the subsamples, and re-estimate the model, imposing this constraint.[13] Twice the difference between the log-likelihoods for the unconstrained and constrained models is asymptotically distributed chi-squared with degrees of freedom equal to the number of parameters constrained. The forward-looking models contain four IS and inflation parameters, plus four

[13] In the estimates reported, we hold the reaction function parameters at their OLS estimates, rather than estimating these parameters jointly with the rest of the model.

autoregressive error parameters for the model with serially correlated errors. The backward-looking model has four IS and four inflation parameters.

The LR tests for the three models across a variety of breakpoints are reported in the table below.

Table 8. Likelihood Ratio Test for Joint Stability of IS, Inflation Specifications
Numbers displayed are *p*-values (Degrees of freedom in parentheses)

	Sample Splits			
Model	(1) 66:1-79:3 (2) 79:4-82:3	(1) 66:1-79:3 (2) 79:4-87:3	(1) 66:1-79:3 (2) 79:4-97:4	Multiple Splits[a]
Backward-Looking	.66 (8)	.46 (8)	.35 (8)	.19 (24)
Forward-Looking	$1.4e^{-9}$ (4)	$2.4e^{-8}$ (4)	$9.0e^{-15}$ (4)	$5.2e^{-23}$ (12)
Forward-Looking, AR(2) errors	.049 (8)	.044 (8)	.0015 (8)	.013 (24)

[a]The "multiple splits" column includes splits based on the breakpoint analysis in section 2 and on prior knowledge of Federal Reserve operating procedures and appointments. The breaks occur at 1979:3, 1982:3, and 1987:2, corresponding to the October 1979 change in operating procedures, the return to conventional operating procedures, and the beginning of Alan Greenspan's chairmanship, respectively.

As the table indicates, the backward-looking model fails to reject for any of the sample splits explored, yielding asymptotic *p*-values of 0.19 or greater in all cases. The forward-looking model that assumes *iid* errors rejects very strongly for all sample breakpoints. The forward-looking model that allows for second order autocorrelation in the errors rejects at the 5 percent level or lower for all breakpoints. The longer the test period after the assumed breakpoint, the more confident the rejection of the forward-looking models. The *p*-value for the backward-looking model also falls as the number of observations after the breakpoint increases.[14]

The strength of the rejection of stability for the forward-looking model with *iid* errors is striking, especially given its poor in-sample fit. A closer examination of the results reveals that the rejection is entirely due to a shift in the intercepts in the equations.

[14] A limited investigation of the finite-sample distribution of the likelihood ratio (LR) test for these models generally found a shift to the right of the LR test relative to the asymptotic chi-squared distribution. However, use of the empirical distribution generally leads to the same qualitative conclusions as the asymptotic distribution: The backward-looking model fails to reject, while the simple forward-looking model rejects decisively. The test of stability for the forward-looking model with AR(2) errors with the longest second sample develops a *p*-value of 0.11. The appendix discusses further test results with empirical finite sample distributions.

That is, a test of the stability of b_1 and c_1, allowing the intercepts b_0 and c_0 to vary across known breakpoints, does not reject at conventional levels. This reaffirms the intuition developed from the graphical information presented above. Other than a very significant shift in the average level of inflation and the output gap, the simple forward-looking model fits so poorly that one cannot reject stability across sub-samples.

The straightforward conclusions to draw from these results are as follows: (1) the full-information approach yields greater power to detect structural shifts by exploiting information about the joint behavior of inflation and output, and (2) both forward-looking models exhibit evidence of instability across regime shifts, while the backward-looking models fail to reject stability in all cases. These formal statistical tests corroborate the pictorial evidence presented in figures 3 to 5.

Likelihood Ratio Tests: Unknown Breakpoints

Here we implement system versions of the single-equation unknown breakpoint tests reported above. We explore two variants of the test. In both of the tests, we examine the likelihood ratio test for stability of some of the model parameters across a single unknown breakpoint. In the first, we test the stability of only the Phillips and IS curve parameters in the models. To do so, we first estimate two reaction functions, one from the beginning of the sample to a particular breakpoint, and one from that breakpoint to the end of the sample. We then estimate the full model holding the reaction function parameters at these split-sample estimates, first constraining the Phillips and IS parameters to be constant across the breakpoint, and then allowing them to vary across the breakpoint. We report twice the difference in the log-likelihood ratios under the unconstrained and constrained cases in the first set of columns in Table 9 below.

As the table indicates, stability of the Phillips and IS parameters is overwhelmingly rejected in both versions of the MN model. The largest value of the test statistic is recorded in the late 1970s for both models, and the *p*-value at these breakpoints is essentially zero. But stability of the RS model's non-policy parameters can also be rejected, although not as strongly. The statistic takes its maximum value in 1983:2, with a *p*-value of about .02. While the *p*-values indicate that the rejections are several orders of magnitude stronger for the MN model, these results still suggest moderate instability in the RS model's PC and IS parameters, conditional on estimated changes in the reaction function parameters.

Table 9. Likelihood Ratio Test for Joint Stability of IS, Inflation, Reaction Function Parameters, Unknown Breakpoints

	Phillips/IS only			All parameters		
Model	Test (df)	Date	*p*-value	Test (df)	Date	*p*-value
Backward-Looking	25.5 (8)	83:2	.019	160.8 (13)	85:1	.000
Forward-Looking	99.4 (4)	76:2	.000	233.6 (9)	85:1	.000
Forward-Looking, AR(2)	101.2 (8)	78:2	.000	156.3 (13)	85:1	.000

In the second test, we jointly test for the stability of the reaction function parameters along with the Phillips Curve and IS parameters. The constrained model holds the reaction function, Phillips, and IS parameters fixed across the entire sample. The unconstrained model allows all parameters to vary across a particular breakpoint. The second set of columns in Table 9 displays the results of this test.

Jointly testing for the shift in the reaction function clearly increases the size of the test for all models, although the increase is largest proportionally for the RS model. Thus, these results provide additional support for the presence of a break in the reaction function somewhere in the late 1970s or early 1980s. They also suggest that *none* of the models can lay claim to stability over the past 35 years, especially once we allow for the possibility of breaks in the reaction function.

5. Conclusions

The Lucas critique is probably both one of the most widely accepted tenets of the economics profession and one of the most frequently ignored in practice. It is widely accepted because of the straightforward and compelling logic of Lucas's (1976) argument. It is frequently ignored in practice because in order to heed its implications fully one must construct a truly structural model, in Lucas's terms, and that is an extraordinarily daunting task.

In this paper, we have shown that a common modeling technique designed to deal with the Lucas critique – the construction of forward-looking rational expectations models based on deep parameters – is no guarantee of success in dealing with the instability problem identified by Lucas. We approach the Lucas critique as the source of an empirically testable hypothesis and we put to the test both forward- and backward-looking models. Perhaps surprisingly, there is little evidence that the backward-looking models are unstable. In principle, the Lucas critique applies most forcefully to these types of models, but in practice the magnitude of the Lucas effect – the reaction of agents' behavioral equations to structural changes in policy – does not seem very large.

In contrast, the forward-looking equations designed to deal with regime changes exhibit clear evidence of instability, especially when subjected to the full-information techniques of section 4. The fact that these equations seem inferior to their backward-looking counterparts in terms of fit probably reduces the ability of statistical tests to reject specific hypotheses. Thus, the fact that the full-information tests still reject stability is all the more significant.

The purpose of this paper, however, is not to argue that the attempt to formulate structural forward-looking models is misguided. We wish instead to emphasize the following implications of our analysis. First, the Lucas critique is not a pure theoretical result, but rather a warning that highlights the importance of applying stability tests to macroeconomic models. In practice, no model is strictly policy invariant and results must be viewed empirically as relative and not as absolute. Second, a corollary of the first conclusion is that every model should be thoroughly tested for stability before being used for policy analysis.

Third, the fact that an empirical model is founded on a theoretical model with optimizing agents and rational expectations does not mean that the model will be empirically stable. Fourth, the fact that a model is backward-looking or is a reduced form does not mean that it will be strongly susceptible to policy changes. Fifth, of the current crop of small macroeconomic models, it seems that backward-looking formulations, which fit the data better, are somewhat more stable than their forward-looking counterparts.

References

Alogoskoufis, George S. and Ron Smith, 1991, "The Phillips Curve, the Persistence of Inflation, and the Lucas Critique," *America Economic Review*, **81(5)**:1254-1275.

Anderson, Gary and George R. Moore, 1985. "A linear algebraic procedure for solving linear perfect foresight models," *Economics Letters*, **17**:247—252.

Andrews, Donald W.K., 1993, "Tests for Parameter Instability and Structural Change with Unknown Change Point," *Econometrica*, **61(4)**:821-856.

Andrews, Donald W.K. and Ray C. Fair, 1988, "Inference in Nonlinear Econometric Models with Structural Change," *Review of Economic Studies*, **55**:615-640.

Berndt, E.K., B.H. Hall, R.E. Hall and J.A. Hausman, 1974, "Estimation and Inference in Nonlinear Structural Models," *Annals of Economic and Social Measurement*, **3(4)**:653-665.

Clarida, Richard, Jordi Gali and Mark Gertler, 1999. "The Science of Monetary Policy," Manuscript, Forthcoming *Journal of Economic Literature*.

Estrella, Arturo and Jeffrey C. Fuhrer, 1998, "Dynamic Inconsistencies: Counterfactual Implications of a Class of Rational Expectations Models," Federal Reserve Bank of Boston Working Paper No. 98-5.

Engle, Robert F. and David F. Hendry, 1993, "Testing Superexogeneity and Invariance in Regression Models," *Journal of Econometrics*, **56(1/2)**:119-139.

Ericsson, Neil R., David F. Hendry and Grayham E. Mizon, 1998, "Exogeneity, Cointegration, and Economic Policy Analysis," *Journal of Business and Economic Statistics*, **16(4)**:370-387.

Ericsson, Neil R. and John S. Irons, 1995, "The Lucas Critique in Practice: Theory Without Measurement," Board of Governors of the Federal Reserve System, International Finance Discussion Paper No. 506.

Favero, Carlo and David F. Hendry, 1992, "Testing the Lucas Critique: A Review," *Econometric Reviews*, **11(3)**:265-306.

Ghysels, Eric, and Alastair Hall, 1990. "A Test for Structural Stability of Euler Conditions Parameters Estimated via the Generalized Method of Moments Estimator," *International Economic Review*, **31(2)**:355-364.

Ghysels, Eric, Alain Guay and Allastair Hall, 1997, "Predictive Tests for Structural Change with Unknown Breakpoint," *Journal of Econometrics*, **92**:209-233.

Hansen, Lars P., 1982, “Large Sample Properties of Generalized Method of Moments Estimators,” *Econometrica*, **50**:1029-1054.

Lucas, Robert E., Jr., 1976. “Econometric policy evaluation: a critique,” *Carnegie-Rochester Conference Series on Public Policy*, 7-33.

McCallum, Bennett T. and Edward Nelson, 1998, AOperational Policy Rules in an Optimizing IS-LM Model,@ Prepared for the joint CEPR/San Francisco Fed conference.

Miller, Preston and William T. Roberds, 1991, “The Quantitative Significance of the Lucas Critique,” *Journal of Business and Economic Statistics*, **9(4)**:361-387.

Newey, W.K. and K. West, 1987, “A Simple Positive Semi-Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix,” *Econometrica*, **55**:703-708.

Oliner, Stephen D., Glenn D. Rudebusch and Daniel Sichel, 1996, “The Lucas Critique Revisited: Assessing the Stability of Empirical Euler Equations for Investment,” *Journal of Econometrics*, **70(1)**:291-316.

Roberts, John. 1995. ANew Keynesian Economics and the Phillips Curve,@ *Journal of Money, Credit, and Banking,* **37:(4)**, 975-984.

Rotemberg, Julio J. and Michael Woodford, 1998, “Interest Rate Rules in an Estimated Sticky-Price Model,” Forthcoming in John B. Taylor, ed., *Monetary Policy Rules,* Chicago: University of Chicago Press.

Rudebusch, Glenn and Lars E.O. Svensson, 1998. "Policy Rules for Inflation Targeting," NBER Working Paper No. 6512 (Forthcoming in John B. Taylor, ed., *Monetary Policy Rules,* Chicago: University of Chicago Press).

Taylor, John B., 1993, “Discretion Versus Policy in Practice,” *Carnegie-Rochester Conference Series on Public Policy,* **39**:195-214.

Appendix 1: Form of the Lagrange multiplier (LM) and predictive (TS) tests

Let the model to be tested be $y_t = f(q, x_t) + u_t$ and assume that the model is estimated by imposing orthogonality conditions of the form $E(g) = 0$, where g is a vector function of the data and the parameters. As in Hansen (1982) and Newey and West (1987), assume that the estimate is obtained by minimizing over q the quadratic form $g'Wg$. Also, let $D = \partial g / \partial q$ and let S represent the Newey-West (1987) matrix of weighted residual autocovariances. Then a consistent estimator of the variance of $\hat{q}$ is $V = (D'WD)^{-1} D'WSWD(D'WD)^{-1}$.

For the purpose of constructing the test statistics, define also $M = (D'WD)^{-1} D'W$. Then the Lagrange multiplier is defined as in Andrews and Fair (1988) as

$$\text{LM} = \frac{1}{p_1 p_2} g_1' M' V^{-1} M g_1,$$

and the Ghysels-Hall (1990) predictive statistic is defined as

$$\text{TS} = g_2' (S_2 + D_2 V_1 D_2')^{-1} g_2,$$

where the subscript i indicates that the component is calculated from data in the ith subsample (but with parameter estimates from the first subsample in the case of TS), and where p_i indicates the proportion of the data in the ith subsample.

Andrews and Fair (1988) and Ghysels and Hall (1990), respectively, show that under standard regularity conditions LM and TS are distributed asymptotically as chi-squared with degrees of freedom equal to the number of parameters that may change across subsamples. In follow-up articles, Andrews (1993) and Ghysels, Guay and Hall (1997) derive the asymptotic distribution of *sup LM* and *sup TS,* where the *sup* is taken over an interior portion of the full sample that excludes some observations at each end. This statistic may be used to test for a break when the break point is unknown. In the text, we provide p values of these tests based on simulations (100,000 iterations) of the asymptotic distribution, which as shown by Andrews (1993) is given by a Bessel process.

Note that in many specific cases, the formulas above simplify considerably. For instance, if the estimates are obtained by least squares, then $g = X'u$ and $D = W^{-1} = X'X$. If instrumental variables are used, then $g = Z'u$, $D = Z'X$, and

$W = (Z'Z)^{-1}$. If the estimates are obtained by maximizing a log-likelihood function L, then $g = \partial L / \partial q$ and, if the disturbance u is iid, Berndt et al. (1974) have shown that $E(D) = E(S)$. Thus, simplification obtains if we assume that, given the expectational equivalence, D=S holds approximately, and that W=I.

Appendix 2: Computation of the Likelihood Function

Model Solution and Observable Representation

Each of the stochastic linear rational expectations models that we consider can be cast in the format

$$\sum_{i=-t}^{0} H_i x_{t+i} + \sum_{i=1}^{q} H_i E_t x_{t+i} = e_t \tag{6.1}$$

where J and 2 are positive integers, x_t is a vector of variables, and the H_i are conformable n-square coefficient matrices, where n is the number of endogenous variables in the model. The coefficient matrices H_i are completely determined by a set of underlying structural parameters 1. The expectations operator $E_t(.)$ denotes mathematical expectations conditioned on the process history through period t,[15]

$$E_t x_{t+i} = E(x_{t+i} | x_t, x_{t-1}, \ldots) . \tag{6.2}$$

The random shock ,$_t$ is independently and identically distributed *N(0,S)*. Note that the covariance matrix S is singular whenever equation (6.1) includes identities. Of little importance are accounting identities such as the national income accounting identity linking GDP and consumption, investment, government expenditures, and net exports. Of more importance are “expectational identities” such as the identity that defines the *ex ante* long-term real interest rate, D, in the pure expectations hypothesis definition of the long rate

$$r_t = \sum_{i=0}^{\infty} b^i E_t (r_{t+i} - p_{t+i+1}) . \tag{6.3}$$

[15] The code for computing the observable structure allows an expectations viewpoint date of either t or t-1. For simplicity, we focus on the t-period expectations case here.

Expectational identities are important because they define variables such as the long real rate that can only be observed within the context of the model.

Because $,_{t}$ is white noise, $E_t(,_{t+k})=0$. Leading equation (6.1) by one or more periods and taking expectations conditioned on period-t information yields a deterministic forward-looking equation in expectations,

$$\sum_{i=-t}^{q} H_i E_t(x_{t+k+i}) = 0, k > 0 \quad . \tag{6.4}$$

We use the Anderson-Moore (1985) procedure to solve equation (6.4) for expectations of the future in terms of expectations of the present and the past. For a given set of initial conditions, $E_t(x_{t+k+i}):k>0,\ i=-J,\ldots,-1$, if equation (6.1) has a unique solution that grows no faster than a given upper bound, that procedure computes the vector autoregressive representation of the solution path,

$$E_t x_{t+k} = \sum_{i=-t}^{-1} B_i E_t(x_{t+k+i}),\ \ k > 0 \quad . \tag{6.5}$$

In the models we consider here, the roots of equation (6.5) lie on or inside the unit circle.

Using the fact that $E_t(x_{t-k})=x_{t-k}$ for $k\$0$, equation (6.5) is used to derive expectations of the future in terms of the realization of the present and the past. These expectations are then substituted into equation (6.1) to derive a representation of the model that we call the *observable structure*,

$$\sum_{i=-t}^{0} S_i x_{t+i} = e_t \quad . \tag{6.6}$$

Equation (6.6) is a structural representation of the model because it is driven by the structural disturbance, $,_t$; the coefficient matrix S_0 contains the contemporaneous relationships among the elements of x_t. It is an observable representation of the model because it does not contain unobservable expectations.

Computing the Likelihood Function

Having obtained the observable structure of equation (6.6), it is relatively straightforward to compute the value of the likelihood function given the data and parameter values. The likelihood is defined as

$$L = T(\log|J| - .5\log\hat{\Omega}) \tag{6.7}$$

where T is the sample size, J is the Jacobian of the transformation from x to g (which is time-invariant by assumption within subsamples), and **S** is the variance-covariance matrix of the structural residuals , $_t$.

Consider concatenating the n x n coefficient matrices S_i, ordered left to right from i=-**J** to 0. We denote this n by n x $(J+1)$ matrix '. Define the vector stack of the endogenous variables at time t as $X_t = [x_{t-J},...,x_t]'$. Thus equation (6.6) may be rewritten

$$\Gamma\ X_t = e_t\ . \qquad (6.8)$$

In computing the value of the likelihood, it will be useful to partition ' as follows. Denote stochastic equations by the subscript s, identity equations by the subscript i, and denote data variables with the subscript d, and "not-data" variables (such as the unobserved long real rate defined above) with the subscript n. Arbitrarily ordering the observable structure so that stochastic equations appear in the top rows and data variables in the left columns of each block, we can write equation (6.6) as

$$\left[\begin{array}{c|cc} S_{s,t-1} & S_{s,d} & S_{s,n} \\ \hline S_{i,t-1} & S_{i,d} & S_{i,n} \end{array}\right]\left[\begin{array}{c} X_{t-1} \\ \hline X_{d,t} \\ X_{n,t} \end{array}\right] = \left[\begin{array}{c} e_t \\ \hline 0 \end{array}\right]. \qquad (6.9)$$

$S_{s,t-1}$ denotes the coefficient block of ' for the lagged variables that enter the stochastic equations; $S_{i,t-1}$ is the corresponding block for identity equations. The right-hand-most n by n block of equations, representing the coefficients on contemporaneous variables, is further partitioned vertically into its data and not-data components.

For each observation t, we use this concatenated, partitioned version of the observable structure to solve for the residuals , $_t$. First, solve for the period-t not-data variables as

$$X_{n,t} = -S_{i,n}^{-1}[S_{i,t-1}X_{t-1} + S_{i,d}X_{d,t}] \qquad (6.10)$$

Now substitute the solution for $X_{n,t}$ into the top rows of equation (6.9) to solve for , $_t$:

$$e_t = S_{s,t-1}X_{t-1} + S_{s,d}X_{d,t} - S_{s,n}S_{i,n}^{-1}[S_{i,t-1}X_{t-1} + S_{i,d}X_{d,t}] \qquad (6.11)$$

The residuals for each time period $t=1,...,T$ are computed, and the residual covariance matrix is then computed as

$$\Omega = (1/T)ee'\ .$$

Note that implicit in the solution for the residuals (equation (6.11)) is the definition of the Jacobian, $\partial\varepsilon/\partial x$,

$$J \equiv S_{s,d} - S_{s,n} S_{i,n}^{-1} S_{i,d} \quad . \tag{6.12}$$

Full Information Maximum Likelihood Estimation

Maximum likelihood estimation consists of finding the parameter values **1**, implicit in the coefficient matrices H_i of equation (6.1), that maximize equation (6.7). We use Matlab's sequential quadratic programming algorithm *constr* to maximize the likelihood function, subject to several types of constraints:

- Parameter boundary constraints (upper and lower bounds for the elements of **1**);
- Equality constraints of the form *F(**1**) = 0*;
- Inequality constraints of the form *G(**1**)#0*. Our routine always enforces the nonlinear inequality constraint that the current parameter setting must be consistent with the correct number of large roots (the number of roots whose magnitude exceeds the specified upper bound is consistent with a unique, stable solution) in a converged solution.

The procedure uses numerical derivatives of the likelihood with respect to the parameters and of the constraints with respect to the parameters. Standard errors are computed from the numerical estimate of the Hessian, $H = \dfrac{\partial L}{\partial\Theta\partial\Theta'}$ as

$$\text{se} = \sqrt{diag(H^{-1})}$$

where *diag* indicates the diagonal elements of the inverse Hessian matrix.