

Assessing the Effects of Large-Scale Asset Purchases in a Zero-Interest-Rate Environment through the Lens of DSGE and VAR Models

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Abstract

Large-scale asset purchases (LSAPs) are ineffective (neutral operations) in standard dynamic stochastic general equilibrium (DSGE) models, and standard DSGE models forecast an increase in interest rates immediately after the recent recession given the predicted output and inflation, contradictory to the extended period of near-zero interest rate policy (ZIRP) conducted by the Federal Reserve. In this paper, I study two mechanisms for breaking LSAPs' neutrality as in Chen, Cúrdia, and Ferrero (2012) and Harrison (2012), and I also study two methods of modeling the ZIRP in DSGE models: the perfect foresight rational expectations model and the Markov regime-switching model which I develop in this paper. In this regime-switching model, I assume that, in one regime, the policy follows a Taylor rule, while, in the other regime, it involves a zero interest rate. I also construct the optimal filter to estimate this regime-switching DSGE model with Bayesian methods. I fit those modified DSGE models to the U.S. data from the third quarter of 1987 to the second quarter of 2010, and then, starting from the third quarter of 2010, I simulate the U.S. economy forward under four environments: no policy intervention, only LSAPs, only ZIRP for an extended period,

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and the combination of LSAPs and ZIRP. I compare the predicted paths of the macro variables under these four scenarios through cross-assessment of the different models. I find that the sole LSAPs intervention has an insignificant effect. The efficacy of the ZIRP crucially depends on the models: The estimated regime-switching model I develop implies a substantial stimulative effect (on average a 12.8% increase in output level and a 2.1% increase in inflation accumulatively over 20 quarters), while the perfect foresight rational expectations model implies a five-fold stronger stimulus to inflation. The actual path from the third quarter of 2010 onward is closer to the predicted path of the regime-switching model. Furthermore, I use VARs that relax the DSGE model restrictions to examine the reason for the small effects of LSAPs measured in the DSGE models. In summary, the regime-switching model I propose is more appropriate to assess the effectiveness of the ZIRP. The ZIRP is effective in stimulating the economy, but the efficacy of LSAPs is uncertain.

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1 Introduction

In response to the 2008-2009 financial crisis, economic recession, and the weak recovery that followed, the Federal Reserve has been giving the economy unprecedented support: the federal funds rate has been kept close to zero since late 2008, and the Federal Reserve has launched three rounds of large-scale asset purchases (LSAPs) (also known as "Quantitative Easing" (QE) by the financial community and financial media). The Federal Reserve purchased a total of \$1.75 trillion in agency debt, mortgage-backed securities, and Treasury notes starting in December 2008, followed by a second \$600 billion Treasury-only program in the fall of 2010. An additional \$400 billion "Operation Twist" program was announced in September of 2011. This program was a pure swap between short-term and long-term assets, and it did not create additional reserves. "QE3" was announced on September 13, 2012. The Federal Reserve has pledged to purchase \$40 billion monthly of agency mortgage-backed securities in an open-ended commitment in hopes of lowering the unemployment rate while maintaining extraordinarily low rate policy, which I refer to as zero interest rate policy (ZIRP), until "at least mid-2015." "QE4" was announced on December 12, 2012. The Federal Reserve is going to continue buying \$40 billion monthly of agency-backed mortgage securities while using \$45 billion monthly created reserves to purchase intermediate and long term Treasury notes until unemployment falls to 6.5% and as long as "inflation between one and two years ahead is projected to be no more than a half percentage point above the committee's 2 percent longer-run goal, and longer-term inflation expectations continue to be well anchored".¹ Bernanke and Reinhart (2004) refer to both the asset purchases and the commitment to keep interests low (forward guidance) as "unconventional monetary policy,"

¹The Bank of England also set up an asset purchases facility in early 2009, and has bought £375 billion assets (\$600 billion) at the time of writing. The European Central Bank purchased €60 billion (\$80 billion) of the Euro area covered bonds (a form of corporate bonds). The bank of Japan has expanded its asset purchases program to a total of ¥55 trillion (\$696 billion).

because conventional monetary policy refers to the manipulation by the central bank of the policy rate, which is the federal funds rate in the United States. Standard DSGE models designed to analyze monetary policy and match the macro data well before the crises must address the challenge of evaluating the Federal Reserve's unconventional policy. There are two main issues.

The first issue is that asset purchases are completely ineffective (neutral operations) in the baseline New Keynesian model of Eggertsson and Woodford (2003). Market participants take full advantage of arbitrage opportunities, thus LSAPs should have no effect on real economic outcomes. The LSAPs' neutrality result only depends on two postulates: All investors can sell and buy the same assets at the same market prices, and assets are only valued for their pecuniary returns. In order for LSAPs to have a real effect, a natural starting point is to break either one of these postulates. Chen, Cúrdia, and Ferrero (2012) introduce financial market segmentation to break the first postulate, which implies that the long-term interest rate matters for aggregate demand distinctly from the expectation of short-term rates. Some households are constrained in the sense that they can only invest the long-term bonds. In this world, asset purchases that successfully reduce the yield on long-term bonds should tilt the consumption profile of the constrained households towards the present and stimulate investment. This will have a positive consequence for both output and inflation. Harrison (2012)² assumes bonds-in-utility to break the second postulate. Since bonds directly enter agents' Euler equation, central banks' asset purchases program affects agents' consumption choice, and thus aggregate output and inflation, by affecting the quantity of outstanding long-term bonds.

The second issue is that since December of 2008, the U.S. federal funds rate has been effectively zero. Standard DSGE models assume a Taylor rule, which often predicts a quick

²Both Chen et al. (2012) and Harrison (2012) are some variation of Andrés et al. (2004).

rise of interest rates immediately after a recession.³ When analyzing the effects of the policy of keeping the interest rates extremely low for an extended period, the standard approach is to estimate a *stochastic* model and then conduct a counterfactual analysis using the perfect foresight rational expectations (PFRE) solution method (Cúrdia and Woodford (2011)).⁴ This method assumes that agents have *perfect foresight* of the path of future shocks and the interest rates, and rational expectations equilibrium can be solved backwards. The policy analysis inherently conflicts with the assumption of the stochastic model that is used to fit the data. Furthermore, the PFRE model predicts an unrealistic path of macro variables. For example, this model predicts a spurious rise in inflation⁵.

In this work, I study two types of DSGE models that break the neutrality of LSAPs as in Chen et al. (2012) and Harrison (2012) and two methods of modeling the ZIRP in DSGE models: the PFRE model and the regime-switching model I develop in this paper in order to better predict the distribution of macroeconomic variables. I fit those DSGE models to the U.S. data from the third quarter of 1987 to the second quarter of 2010, and then, starting from the third quarter of 2010, I simulate the U.S. economy forward under four scenarios: the counterfactual scenario when there is no policy intervention, only LSAPs intervention, only ZIRP for an extended period, and the combination of LSAPs and ZIRP. In order to assess the effectiveness of the asset purchases policy and the policy of an extended period of near-zero interest rates, I compare the predicted path of the macro variables (output and inflation) under the policy intervention with the predicted path of the macro variables absent of both asset purchase and ZIRP (the counterfactual scenario when there is no policy intervention). I found that the effects of the LSAPs alone are insignificant measured in the DSGE models, while the ZIRP has a substantial effect.

³Reichschneider and Williams (2000), Chung et al. (2011), and Del Negro and Schorfheide (2012).

⁴A detailed description can be found at the online appendix of Chen, Cúrdia and Ferrero (2012).

⁵Carlstrom, Fuerst, and Paustian (2012) interpret the explosive dynamics as a failure of New Keynesian monetary DSGE models, and Blake (2012) shares this sentiment.

In Chen et al. (2012) the ZIRP is modeled by the PFRE model. This paper proposes to model the ZIRP by a regime-switching monetary policy rule where, in one regime, the policy rates follow a typical Taylor rule, and, in the other regime, it involves a policy of zero interest rates. I solve this regime-switching DSGE model by using the Farmer, Waggoner, and Zha (2011) minimum state variable solution. I construct the optimal filters in order to estimate this regime-switching DSGE model with Bayesian methods. I compare this method of modeling the ZIRP in DSGE models with the PFRE. The simulation of the Federal Reserve's ZIRP reveals that the effects of ZIRP on macro variables crucially depend on the models: the regime-switching model implies a substantial effect of ZIRP while PFRE implies a five-fold stronger stimulus of ZIRP to inflation. The fundamental difference between these two types of models is how agents' expectations are formulated. In the regime-switching model, at each period agents attach certain probability of exiting the ZIRP regime in the next period despite the Federal Reserve's "extended period" language, because, for example, the simple announcement would be subject to the time inconsistency problem, and is thus incredible. The PFRE assumes that agents believe the Federal Reserve's announcement and have perfect foresight of future interest rates. The predicted path of macro variables generated by the regime-switching model is closer to the actual path.

Here, I am looking at this extended period of zero interest rates as a policy choice⁶ because the central bank could raise the interest rates when the output starts growing, and the economy is improving as advised by the Taylor rule. Carlstrom, Fuerst, and Paustian (2012), Cúrdia and Woodford (2010), and Chen, Cúrdia, and Ferrero (2012) also study the effects of a *transient* interest rate peg. Under the assumption of either a deterministic exit or a stochastic exit of the interest rate peg in the previous studies, the policy rate will follow a Taylor rule after the exit and the interest rate peg will never occur again. In my

⁶Here the regime-switching is exogenous while ideally it should be endogenous and depend on the macro-economic condition.

regime-switching model, however, zero interest rate policy regime is a recurring event. Even at the normal interest rate regime, agents expect to enter zero interest rate regime in the future with certain probability. Expectations play an important role in the regime-switching model. An alternative angle to look at this persistent period of low interest rates is the zero lower bound (ZLB) problem. A persistent shock⁷ drives interest rates below zero if we keep following a Taylor rule. A rapidly growing literature on ZLB considers the zero interest rates as a modeling constraint that has to be considered. Global methods include Judd, Maliar, and Maliar (2011), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012), and Aruoba and Schorfheide (2012). There are also a few short cuts for modeling ZLB: such as Braun and Körber (2011), Adam and Billi (2007), Eggertsson and Woodford (2003), and Christiano, Eichenbaum, and Rebelo (2011). Del Negro and Schorfheide (2012) describe how to impose zero interest rates via unanticipated or anticipated monetary policy shocks in a DSGE model.

DSGE models impose strict cross-equation restrictions. I use VARs that relax the DSGE model restrictions to further examine the reason for the small effects of LSAPs measured in the DSGE models. I investigate how the effects of LSAPII are empirically identified in the DSGE models that break the neutrality of the LSAP operation such as Chen et al. (2012) and Harrison (2012). I ask the questions: What happens when you relax some of the DSGE model restrictions? How do DSGE models compare to VAR studies? Using the exogenous restrictions implied by the DSGE models, the estimated VAR model suggests no evidence of positive effects of LSAP on output and inflation. An estimated VAR with a further relaxation of DSGE restrictions can generate a sizable effect of LSAPs but with considerable uncertainty.

The rest of the paper proceeds as follows. The next section presents two types of models where I describe how the LSAPs' neutrality result can be broken in the DSGE models.

⁷For example a preference shock or a technology shock.

Section 3 discusses how to model ZIRP with a regime-switching monetary policy and with the PFRE. Section 4 describes the estimation of the regime-switching model, some basic analysis of parameter estimates, an evaluation of the effects of the LSAPs and the ZIRP, and the comparison between the regime-switching model and the PFRE model. I discuss the identification of the LSAPs in the DSGE models and the comparison with VARs in Section 5. Finally, section 6 concludes.

2 Models

In the households sector, I will explain how the typical no-arbitrage condition for short-term and long-term bonds can be broken in order for LSAPII to have a real effect. I will describe two models: Chen et al. (2012) and a variation of Harrison (2012). The rest of the sectors are standard in medium-scale DSGE models (Christiano et al. (2005); Smets and Wouters (2007)): Monopolistic competitive firms hire the labor to produce intermediate goods; competitive final goods producing firms package intermediate goods into a homogeneous consumption good. Finally, the government sets monetary and fiscal policy. To simplify the analysis, I abstract from capital.

2.1 Households

A common means by which the asset purchases are effective is that if the central bank changes its portfolio composition in equilibrium, private investor must also change their portfolio choices, and, in order to induce them to do so, the equilibrium asset prices must also change accordingly. However, a mere difference in state-contingent returns on different assets is not enough for central bank portfolio changes to have an effect because the private investors will fully take advantage of the arbitrage opportunities and hedge against the central bank's operation. Cúrdia and Woodford (2011) present a detailed explanation for

this. This neutrality result only depends on two postulates: All investors can buy or sell the same assets at the same market prices, and all assets are valued only for their pecuniary returns. Chen et al. (2012) propose market segmentation to break the first postulate while Harrison (2012) targets the second postulate. Both approaches are based on Andrés et al. (2004). Throughout the paper, I will refer to the first approach as "market segmentation" approach and the second as the "BIU" (bonds-in-utility) approach.

2.1.1 Market Segmentation

To keep the paper self-contained, I briefly reproduce the household sector of the model with slight simplification⁸. For a detailed description, please refer to Chen et al. (2012). The key modification of Chen et al. (2012) relative to a standard medium-scale DSGE model along the lines of Christiano et al. (2005) and Smets and Wouters (2007) is the introduction of segmentation and transaction costs in bond markets, as in Andrés et al. (2004).

A continuum of measure one of households populates the economy. There are two types of households: unconstrained and constrained households, and two types of bonds exist: short-term and long-term bonds. Constrained households can only invest in long-term bonds, while unconstrained households can invest in both short-term and long-term bonds. Intuitively, some institutions such as pension funds can only invest in certain assets due to financial regulations, while some other institutions can arbitrage between different assets. a household of type $j = u, r$ enjoys consumption C_t^j (relative to productivity Z_t , as in An and Schorfheide (2007)) and dislikes hours worked L_t^j .⁹ Households supply differentiated labor inputs indexed by i , but perfectly share the consumption risk within each group. The life-time utility

⁸To make this model comparable to the bonds-in-utility model with regime switching policy rule, I abstract from consumption habit, because solving and estimating a regime-switching model of this scale is computationally challenging. This is the reason I abstract from capital as well.

⁹Chen et al. (2012) express utility as a function of de-trended consumption as An and Schorfheide (2007) to ensure the existence of a balanced growth path with constant relative risk aversion preferences.

function for a generic household j is

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta_j^s b_{t+s}^j \left[\frac{1}{1 - \sigma_j} \left(\frac{C_{t+s}^j}{Z_{t+s}} \right)^{1 - \sigma_j} - \frac{\varphi_{t+s}^j (L_{t+s}^j(i))^{1 + \nu}}{1 + \nu} \right], \quad (2.1)$$

where $\beta_j \in (0, 1)$ is the individual discount factor, b_t^j is a preference shock, $\sigma_j > 0$ is the coefficient of relative risk aversion, $\nu \geq 0$ is the inverse elasticity of labor supply, and φ_t^j is a labor supply shock. The preference and labor supply shocks both follow stationary AR(1) processes in logs.

Short-term bonds, B_t , are one-period securities purchased at time t that pay a nominal return, R_t , at time $t + 1$. Following Woodford (2001), long-term bonds are perpetuities that cost $P_{L,t}$ at time t and pay an exponentially decaying coupon, κ^s , at time $t + s + 1$, for $\kappa \in (0, 1]$.¹⁰ Price in period t of a bond issued s periods ago, $P_{L,t}(s)$, is a function of the coupon and the current price:

$$P_{L,t}(s) = \kappa^s P_{L,t},$$

and one can deduce that

$$P_{L,t} = \frac{1}{R_{L,t} - \kappa}.$$

This means that one bond that was issued s periods ago is equivalent to κ^s new bonds. This allows us to rewrite the flow budget constraint and only keep track of the stock of total long term debt, B_t^L , rather than the current period's purchases of long-term debt.

$$B_{t-1}^L = \sum_{s=1}^{\infty} \kappa^{s-1} B_{t-s}^L$$

The total population consists of a fraction ω_u of unrestricted households who can trade both short-term and long-term government bonds. However, for each unit of long-term

¹⁰If $\kappa = 1$, this security is a consol.

bonds purchased, unrestricted households have to pay a transaction cost ζ_t to a financial intermediary. The financial intermediary rebates its profits, whose per-capita nominal value is \mathcal{P}_t^{fi} , to the households (regardless of type). The remaining $\omega_r = 1 - \omega_u$ fraction of the population are restricted households who can only invest in long-term bonds, but do not pay transaction costs.

The budget constraint for an unrestricted household is:

$$P_t C_t^u + B_t^u + (1 + \zeta_t) P_{L,t} B_t^{L,u} \leq R_{t-1} B_{t-1}^u + P_{L,t} R_{L,t} B_{t-1}^{L,u} + W_t(i) L_t^u(i) + \mathcal{P}_t + \mathcal{P}_t^{fi} - T_t^u. \quad (2.2)$$

For a restricted household, the budget constraint is:

$$P_t C_t^r + P_{L,t} B_t^{L,r} \leq P_{L,t} R_{L,t} B_{t-1}^{L,r} + W_t(i) L_t^r(i) + \mathcal{P}_t + \mathcal{P}_t^{fi} - T_t^r. \quad (2.3)$$

In equations (2.2) and (2.3), P_t is the price of the final consumption good, $W_t(i)$ is the competitive wage, \mathcal{P}_t are the profits distributed by the intermediate goods producers, and T_t^j are lump-sum taxes.¹¹

Let $\Xi_t^{p,u}$ and $\Xi_t^{p,r}$ represent the Lagrange multipliers for (2.2) and (2.3) respectively. The Euler equations for the unrestricted households with respect to the bond choices are

$$\begin{aligned} \text{Short-term bond:} \quad & \Xi_t^{p,u} = \beta_u \mathbb{E}_t [\Xi_{t+1}^{p,u} R_t], \\ \text{Long-term bond:} \quad & \Xi_t^{p,u} = \beta_u \mathbb{E}_t \left[\Xi_{t+1}^{p,u} \frac{P_{L,t+1} R_{L,t+1}}{(1 + \zeta_t) P_{L,t}} \right], \end{aligned}$$

where the transaction cost, ζ_t , is modelled as a function of the long-term bonds as follows:

$$\zeta_t = \zeta \left(\frac{P_{L,t} B_{z,t}^L}{B_{z,t}}, \varepsilon_{\zeta,t} \right),$$

¹¹Each household receives the same dividend from intermediate goods and pays the same amount of lump-sum taxes.

where $B_{z,t}^L = B_t^L / (P_t Z_t)$, and $B_{z,t} = B_t / (P_t Z_t)$. Chen et al. (2012) do not take a stand on the functional form of $\zeta(\cdot)$. They only assume its first derivative to be positive when evaluated at the steady state. In other words, $\zeta\left(\frac{P_L B_z^L}{B_z}, 0\right) > 0$, and $\zeta'\left(\frac{P_L B_z^L}{B_z}, 0\right) > 0$.

The unrestricted households can arbitrage between the two bonds, subject to the transaction costs. One can show that the risk premium of long-term bonds is a function of the current and future transaction costs. Asset purchases alter the quantity of the long-term bonds supplied to the private sector and thus the risk premium of the long-term bonds.

The Euler equation for the restricted households with respect to the long-term bonds is:

$$\text{Long-term bond: } \Xi_t^{p,r} = \beta_r \mathbb{E}_t \left[\Xi_{t+1}^{p,r} \frac{P_{L,t+1} R_{L,t+1}}{P_{L,t}} \right].$$

It is clear from the restricted households' Euler equation that due to their inability to arbitrage between the short-term bonds and long-term bonds, the change of the long-term bond rates will alter their consumption profile and thus aggregate consumption. LSAPs, by construction, will have a real effect.

2.1.2 Bonds-in-Utility

The representative household's objective function is a slight modification of Harrison (2012):

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s b_{t+s} \left[\frac{\left(\frac{C_{t+s}}{Z_{t+s}}\right)^{1-\sigma}}{1-\sigma} - \frac{\varphi_{t+s} L_{t+s}^{1+\nu}}{1+\nu} - \frac{\tilde{\nu}}{2} \left(\delta \frac{B_{t+s}}{P_{L,t+s} B_{L,t+s}} - 1 \right)^2 \right],$$

where in the last term $\frac{B_{t+s}}{P_{L,t+s} B_{L,t+s}}$ represents the ratio of the market value of short-term bonds to that of long-term bonds. δ is the inverse of the steady state of this ratio so that at steady state, the last term is zero. $\tilde{\nu}$ controls the elasticity of the households' portfolio choice in response to the long-term bond rate. The intuition of bonds-in-utility is similar to

money-in-utility. Because long-term bonds are not as liquid as short-term bonds, holding a non-optimal portfolio composition induces a utility cost.

The time t budget constraint for a household is

$$P_t C_t + B_t + (1 + \zeta_t) P_{L,t} B_t^L \leq R_{t-1} B_{t-1} + P_{L,t} R_{L,t} B_{t-1}^L + W_t L_t + \mathcal{P}_t + \mathcal{P}_t^{f^i} - T_t, \quad (2.4)$$

where, ζ_t , is also a transaction cost (but *not* a function of the bonds) with a nonzero steady state. This is to capture that, at steady state, the yield of the long-term bonds is higher than that of the short-term bonds, as observed in the data. The definitions of the rest of the variables are the same as the market segmentation model described in the previous section.

Let Ξ_t^P represent the Lagrange multiplier for (2.4). The loglinearized Euler equation for the short-term bonds is

$$\frac{\tilde{\nu}}{\Xi B_z} \widehat{BLMV} B_t - \hat{\Xi}_t + \hat{R}_t + \hat{\Xi}_{t+1} - \hat{z}_{t+1} - \hat{\Pi}_{t+1} = 0,$$

where $\widehat{BLMV} B_t = \frac{\frac{B_{z,t}^L}{(R_{L,t} - \kappa)}}{B_{z,t}}$ ¹², and $\widehat{BLMV} B_t = \hat{B}_{z,t}^L - \hat{B}_{z,t} - \frac{R_L}{(R_L - \kappa)} \hat{R}_{L,t}$.

and the loglinearized Euler equation for the long-term bonds is

$$\frac{\tilde{\nu}}{\delta(1 + \zeta) \Xi B_z} \widehat{BLMV} B_t + \hat{\Xi}_t + \hat{\zeta}_t - \frac{R_L}{R_L - \kappa} \hat{R}_{L,t} + \mathbb{E}_t \left[\frac{\kappa}{R_L - \kappa} \hat{R}_{L,t+1} - \hat{\Xi}_{t+1} + \hat{z}_{t+1} + \hat{\Pi}_{t+1} \right] = 0$$

The BIU specification distinguishably differs from the market segmentation approach by allowing the portfolio choice to directly affect the households' consumption choice. This, in turn, will affect the stochastic discount factor and thus the price of the long-term bond. Again, LSAPs are designed to have a real effect. The advantage of this specification is its simplicity. Household heterogeneity dramatically increases the scale of the market segmenta-

¹² $\widehat{B}_{z,t}^L = \frac{B_t^L}{P_t Z_t}$, and $B_{z,t} = \frac{B_t}{P_t Z_t}$

tion model, and thus estimating and drawing from the posterior of the market segmentation model are challenging, while the BIU specification is a lot more manageable.

2.2 Final Goods Producers

The final good, Y_t , is a composite made of a continuum of intermediate goods indexed by $i \in (0, 1)$

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\lambda_f}} di \right]^{1+\lambda_f}. \quad (2.5)$$

The final goods producers buy the intermediate goods on the market, package to Y_t , and sell it to consumers. These firms maximize profits in a perfectly competitive environment. Their problem is:

$$\begin{aligned} \max_{Y_t, Y_t(i)} \quad & P_t Y_t - \int_0^1 P_t(i) Y_t(i) di, \\ \text{s.t.} \quad & Y_t = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\lambda_f}} di \right]^{1+\lambda_f} (\mu_{f,t}). \end{aligned} \quad (2.6)$$

From the first order conditions:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\frac{1+\lambda_f}{\lambda_f}} Y_t.$$

Combining this condition with the zero profit condition, I obtain the expression for the price of the composite final good:

$$P_t = \left[\int_0^1 P_t(i)^{-\frac{1}{\lambda_f}} di \right]^{-\lambda_f}. \quad (2.7)$$

2.3 Intermediate goods producers

Intermediate goods producer i uses the following technology:

$$Y_t(i) = Z_t L_t, \quad (2.8)$$

where Z_t is the technology, and L_t is labor input. The logarithm of the growth rate of productivity, $z_t = \log \left(\frac{Z_t/Z_{t-1}}{\exp(\gamma)} \right)$, follows an AR(1) process:

$$z_t = \rho_z z_{t-1} + \epsilon_{z,t}, \quad \epsilon_{z,t} \sim \mathbb{N}(0, \sigma_{\epsilon_z}^2).$$

Prices are sticky à la Calvo (1983). Specifically, each firm can readjust prices with a probability $1 - \zeta_p$ in each period. For those firms that cannot adjust prices, $P_t(i)$ will increase at the steady state rate of inflation π . For those firms that can adjust prices, the problem is to choose a price level, $\tilde{P}_t(i)$, that maximizes the sum of the expected discounted profits in all states of the future where the firm is stuck with that price:

$$\max_{\tilde{P}_t(i)} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta \zeta_p)^s \Xi_{t+s} \left(\left(\frac{\tilde{P}_t(i) \Pi^s}{P_{t+s}} \right) - w_{z,t+s} \right) \left(\frac{\tilde{P}_t(i) \Pi^s}{P_{t+s}} \right)^{-\frac{1+\lambda_f}{\lambda_f}} Y_{z,t+s},$$

where $\Xi_{t+s} = \Xi_{t+s}^p P_{t+s} Z_{t+s}$, $w_{z,t+s} = \frac{w_{t+s}}{P_{t+s} Z_{t+s}}$, and $Y_{z,t+s} = \frac{Y_{t+s}}{Z_{t+s}}$.

The first order condition for the firm is

$$\begin{aligned} 0 &= \tilde{P}_t(i) \mathbb{E}_t \sum_{s=0}^{\infty} (\beta \zeta_p)^s \Xi_{t+s} \frac{1}{\lambda_f} \left(\frac{\Pi^s}{P_{t+s}} \right)^{-\frac{1}{\lambda_f}} Y_{z,t+s} \\ &\quad - \mathbb{E}_t \sum_{s=0}^{\infty} (\beta \zeta_p)^s \Xi_{t+s} \frac{1+\lambda_f}{\lambda_f} w_{z,t+s} \left(\frac{\Pi^s}{P_{t+s}} \right)^{-\frac{1+\lambda_f}{\lambda_f}} Y_{z,t+s}. \end{aligned} \tag{2.9}$$

Note that all firms that can readjust prices face an identical problem. I will only consider the symmetric equilibrium in which all firms that can readjust prices will choose the same price, so I can drop the i index. From 2.7 it follows that:

$$P_t = \left[(1 - \zeta_p) \tilde{P}_t^{-\frac{1}{\lambda_f}} + \zeta_p [\Pi P_{t-1}]^{-\frac{1}{\lambda_f}} \right]^{-\lambda_f}. \tag{2.10}$$

So

$$1 = (1 - \zeta_p) \left(\frac{\tilde{P}_t}{P_t} \right)^{-\frac{1}{\lambda_f}} + \zeta_p \left[\frac{\Pi}{\Pi_t} \right]^{-\frac{1}{\lambda_f}}.$$

2.4 Government Policies

The monetary policy is taken from Chen et al. (2012). The central bank follows a conventional feedback interest rate rule similar to Taylor (1993), modified to include the interest rate smoothing (Clarida et al., 2000) and to use the growth rate of output instead of the output gap (Justiniano et al., 2011):

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\rho_m} \left[\left(\frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \left(\frac{Y_t/Y_{t-4}}{e^{4\gamma}} \right)^{\phi_y} \right]^{1-\rho_m} e^{\epsilon_{m,t}}, \quad (2.11)$$

where $\Pi_t \equiv P_t/P_{t-1}$ is the inflation rate, $\rho_m \in (0, 1)$, $\phi_\pi > 1$, $\phi_y \geq 0$, and $\epsilon_{m,t}$ is an i.i.d. innovation.¹³ In the section (3.1), I will elaborate how to modify the monetary policy rule to assess ZIRP.

The presence of long-term bonds modifies the standard government budget constraint as follows:

$$B_t + P_{L,t} B_t^L = R_{t-1,t} B_{t-1} + (1 + \kappa P_{L,t}) B_{t-1}^L + P_t G_t - T_t. \quad (2.12)$$

The left-hand side of expression (2.12) is the market value, in nominal terms, of the total amount of bonds (short-term and long-term) issued by the government at time t . The right-hand side is the total deficit at time t , that is, market value plus interest payment of the bonds maturing in that period plus spending G_t net of taxes.

I assume that the supply of the government bonds is exogenous, and the ratio of the market value of long-term bonds to that of the short-term bonds follows a simple autoregressive

¹³Chen et al. (2012) use the output growth in the Taylor rule, instead of the output gap, to avoid the complication of solving and estimating the system characterizing the flexible price equilibrium. In practice, GDP growth relative to trend is often cited as one of the main indicators of real activity for the conduct of monetary policy.

rule

$$\frac{P_{L,t}B_t^L}{B_t} = S \left(\frac{P_{L,t-1}B_{t-1}^L}{B_{t-1}} \right)^{\rho_B} e^{\epsilon_{B,t}}, \quad (2.13)$$

where $\rho_B \in (0, 1)$, and $\epsilon_{B,t}$ is an i.i.d. exogenous supply shock. S is whatever constant needed to make the above equation an identity at the steady state. I interpret LSAPs program as shocks to the ratio of outstanding government long-term liabilities to short-term liabilities compared to the historical behavior of these series.

2.5 Exogenous Processes

The model is supposed to be fitted to data on output, inflation, hours worked, wages, nominal interest rates, and market value of bonds. There are seven structural shocks in total. The logarithm of the technology follows a random walk with drift.

$$\ln Z_t = \gamma + \ln Z_{t-1} + z_t,$$

where the shock z_t follows a first order autoregressive process (AR(1)):

$$z_t = \rho_z z_t + \epsilon_{z,t}.$$

The preference shock to leisure follows an AR(1) process:

$$\ln \varphi_t = \rho_\varphi \ln \varphi_{t-1} + \epsilon_{\varphi,t}.$$

The shock to the discount factor β (intertemporal preference shifter) is also assumed to follow an AR(1) process:

$$\ln b_t = \rho_b \ln b_{t-1} + \epsilon_{b,t}.$$

The government spending is assumed to be an exogenous process:

$$\ln g_t = \rho_g \ln g_{t-1} + \epsilon_{g,t}.$$

The risk premium shock also follows an AR(1) process:

$$\zeta_t = \rho_\zeta \zeta_{t-1} + \varepsilon_{\zeta,t}.$$

The monetary policy shock $\epsilon_{m,t}$ and the bond supply shock $\epsilon_{B,t}$ are independent and identically distributed shocks.

3 Zero Interest Rate Policy

In this section, I describe two methods of studying the effects of ZIRP in DSGE models. Both solution methods take some shortcuts rather than solve fully a nonlinear New Keynesian model incorporating ZIRP. I am going to consider a regime-switching model where, in one regime, the policy rate follows a typical Taylor rule, and, in the other regime, it simply involves ZIRP. Although the regime switching is imposed to the monetary policy rule before loglinearizing the system, the model is a forward-looking Markov-switching linear rational expectations model. Ideally, I should apply the perturbation method for Markov-switching models proposed by Foerster et al. (2011). This method begins from first principles rather than add Markov switching after linearizing the model, and it also allows higher order solutions. Simplifying assumptions in my model may miss some nonlinear interactions between the zero interest rates and the policy functions of the agents, however, I substantially gain tractability. I also construct the optimal filter so that I can fit this model to the macro data including the recent time where the interest rates are maintained near zero for an extended period. This regime-switching model can not only explain the interest rate data, but also pro-

vides a plausible explanation for exiting the zero interest rate policy. This regime-switching model offers a tool to conduct forecasts and counterfactual analysis. The other approach to assessing the ZIRP, PFRE, on the other hand, can not explain the recent episodes of near-zero interest rates. It only asks the counterfactual questions such as what are the effects to the macro variables if the interest rates are kept at zero for an extended period, and agents have perfect knowledge of this policy experiment? Now I define the regime-switching model more precisely .

3.1 Regime-Switching Policy Rule

In this section, I introduce a regime-switching monetary policy rule that will be incorporated into the DSGE models introduced in section 2. I will use the Farmer, Waggoner and Zha (2011) minimum state variable solution method to solve this regime-switching model, and the estimation strategy will be described in section 4.

Consider a regime-switching policy rule where, in one regime, the federal funds rate follows a Taylor rule while, in the other regime, it simply involves the zero interest rates. The policy rule is

$$R_t = (R_t^*(K_t))^{1-\rho_R(K_t)} \left[\left(\frac{\pi_t r}{R_t^*(K_t)} \right)^{\varphi_\pi(K_t)} \left(\frac{Y_t/Y_{t-4}}{e^{4\gamma}} \right)^{\varphi_y(K_t)} \right]^{(1-\rho_R(K_t))} R_{t-1}^{\rho_R(K_t)} \exp(\varepsilon_{R,t}). \quad (3.1)$$

where all the parameters denoted by (K_t) are regime dependent, and R_t^* are the desired regime-dependent target nominal interest rates. Let $K_t = 1$ denote the normal regime, and $K_t = 2$ denote the ZIRP regime. For example, I can set $R_t^*(K_t = 1) = R_1^* = 1.005$ which corresponds to a target 2% annual interest rate at the normal regime, and set $R_t^*(K_t = 2) = R_2^* = 1.0005$ which corresponds to a target 20 basis points annual interest rate at the second

regime. To study the ZIRP, I set

$$\begin{aligned}
R_2^* &= 1, \\
\rho_R(K_t = 2) &= 0, \\
\varphi_\pi(K_t = 2) &= 0, \\
\varphi_y(K_t = 2) &= 0, \\
\sigma_{\varepsilon_{R,t}}(K_t = 2) &= 0.
\end{aligned}$$

I define the ergodic mean of the logarithm of the steady state interest rates as

$$\log(R) = \bar{\lambda}_1 \log(R_1^*) + \bar{\lambda}_2 \log(R_2^*),$$

where $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are ergodic probabilities.

Divide 3.1 by its ergodic mean, R , and thus:

$$\frac{R_t}{R} = \left(\frac{R_t^*}{R}\right)^{(1-\rho_R(K_t))(1-\varphi_\pi(K_t))} \left[\left(\frac{\pi_t}{\pi}\right)^{\varphi_\pi(K_t)} \left(\frac{Y_t/Y_{t-4}}{e^{4\gamma}}\right)^{\varphi_y(K_t)}\right]^{(1-\rho_R(K_t))} \left(\frac{R_{t-1}}{R}\right)^{\rho_R(K_t)} \exp \varepsilon_{R,t}. \quad (3.2)$$

Loglinearize 3.2 and thus:

$$\begin{aligned}
\hat{R}_t &= \rho_R(K_t) \hat{R}_{t-1} + (1 - \rho_R(K_t)) \left[\varphi_\pi(K_t) \hat{\pi}_t + \varphi_y(K_t) \left(\hat{y}_t - \hat{y}_{t-4} + \sum_{i=0}^{i=3} z_{t-i} \right) \right] \\
&\quad + \varepsilon_{R,t} + (1 - \rho_R(K_t)) (1 - \varphi_\pi(K_t)) \hat{R}_t^*
\end{aligned} \quad (3.3)$$

where the last term represents a regime-switching constant. The Farmer, Waggoner, and Zha (2011) minimum state variable solution method does not deal with a system with a constant. I am going to apply the trick by Liu, Waggoner and Zha (2011). They solve a

system where the only regime-switching coefficient is the constant. I can rewrite 3.3 as

$$\begin{aligned}\hat{R}_t = & \rho_R(K_t) \hat{R}_{t-1} + (1 - \rho_R(K_t)) \left[\varphi_\pi(K_t) \hat{\pi}_t + \varphi_y(K_t) \left(\hat{y}_t - \hat{y}_{t-4} + \sum_{i=0}^{i=3} z_{t-i} \right) \right] + \varepsilon_{R,t} \\ & + (1 - \rho_R(K_t)) (1 - \varphi_\pi(K_t)) \left[\log \left(\frac{R_1}{R} \right), \log \left(\frac{R_2}{R} \right) \right] \hat{\mathbf{e}}_{s,t},\end{aligned}$$

where $\hat{\mathbf{e}}_{s,t} = \mathbf{e}_{s,t} - \bar{\mathbf{e}}_s$, and $\bar{\mathbf{e}}_s$ is the ergodic probability. $\mathbf{e}_{s,t}$ is defined as:

$$\mathbf{e}_{s,t} = \begin{bmatrix} \mathbf{1}_{S_t=1} \\ \mathbf{1}_{S_t=2} \end{bmatrix},$$

with $\mathbf{1}_{\{s_t = j\}} = 1$ if $s_t = j$, and 0 otherwise. As shown in Hamilton (1994), the random vector $\mathbf{e}_{s,t}$ follows an AR(1) process:

$$\mathbf{e}_{s,t} = P\mathbf{e}_{s,t-1} + \nu_t, \tag{3.4}$$

where P is the transition matrix of the Markov switching process, and the innovation vector has the property that $\mathbb{E}_{t-1}\nu_t = 0$. In the steady state, $\nu_t = 0$ so that 3.4 defines the ergodic probabilities for the Markov process $\bar{\mathbf{e}}_s$. Schorfheide (2005) also proposes an algorithm to solve DSGE models with a regime-switching constant in the policy rule. One can prove that Schorfheide (2005) and Liu, Waggoner and Zha (2011) give rise to the same solution¹⁴.

By adding two extra variables $\mathbf{e}_{s,t}$, I can use Farmer, Waggoner and Zha (2011) minimum state variable solution to solve this regime-switching model. The solution of the model can be represented by

¹⁴See the appendix for proof.

$$\begin{aligned} Z_t &= G_t(K_t) Z_{t-1} + R_t(K_t) \varepsilon_t \\ \begin{bmatrix} Z_{1,t} \\ Z_{2,t} \end{bmatrix} &= \begin{bmatrix} G_{11} & G_{12} \\ 0 & P \end{bmatrix} \begin{bmatrix} Z_{1,t-1} \\ Z_{2,t-1} \end{bmatrix} + \begin{bmatrix} R_{11} & R_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix} \end{aligned}$$

where I can partition the variables Z_t and the shocks ε_t into two parts, where $Z_{2,t}$ are $[\hat{e}_t(1) \ \hat{e}_t(2)]'$, $\varepsilon_{2,t}$ are $[v_{1,t} \ v_{2,t}]'$, $Z_{1,t}$ are the rest of the states, and $\varepsilon_{1,t}$ are the structural shocks of the DSGE models. I define

$$C(K_t) = G_{12} [\hat{e}_{t-1}(1) \ \hat{e}_{t-1}(2)]' + R_{12} [v_{1,t} \ v_{2,t}]'.$$

Notice that $C(K_t)$ is a regime-dependent constant. Finally I can rewrite the system as follows with regime-switching coefficients:

$$Z_t = C(K_t) + G_t(K_t) Z_{t-1} + R_t(K_t) \varepsilon_t$$

3.2 Model ZIRP by the PFRE

The solution method of the PFRE model was proposed by Cúrdia and Woodford (2011). For a detailed description of the algorithm and an application, please refer to Chen et al. (2012) and its companion online appendix¹⁵. The basic idea is that agents have perfect foresight of the path of the future interest rates and of all shocks until an arbitrary time point. From this point forward all the shocks are zero, and the solution method is standard such as Sims (2002). The system can be solved backwards from this point. The following is a very simple example to illustrate the solution method. Consider the equilibrium system:

¹⁵The appendix can be found at <http://onlinelibrary.wiley.com/doi/10.1111/j.1468-0297.2012.02549.x/supinfo>

$$\hat{y}_t = \mathbb{E}_t [\hat{y}_{t+1}] - \sigma^{-1} (\hat{i}_t - \mathbb{E}_t [\hat{\pi}_{t+1}])$$

$$\hat{\pi}_t = \beta \mathbb{E}_t [\hat{\pi}_{t+1}] + \kappa \hat{y}_t$$

$$\begin{aligned} \hat{i}_t &= \phi_\pi \hat{\pi}_t + \nu_t \quad \text{for } t > K \quad \nu_t = 0 \\ &= 0 \quad \text{for } t = 1, \dots, K-1, K \end{aligned}$$

The solution for $t > K$ is:

$$\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \hat{i}_t \end{bmatrix} = \begin{bmatrix} \psi_{y\nu} \\ \psi_{\pi\nu} \\ \psi_{i\nu} \end{bmatrix} \nu_t$$

The system can be broken into the forward-looking and the backward-looking parts. The

forward-looking part is:

$$\begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \mathbb{E}_t [\hat{y}_{t+1}] \\ \mathbb{E}_t [\hat{\pi}_{t+1}] \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} & 0 \\ -\kappa & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{i}_t \\ \hat{\pi}_t \end{bmatrix}$$

and the backward-looking part is:

$$\begin{bmatrix} 0 & 1 & -\phi_\pi \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{i}_t \\ \hat{\pi}_t \end{bmatrix} = \nu_t$$

At $t = K$, plug in the solution to the forward looking part and thus:

$$\begin{bmatrix} 1 & \sigma^{-1} \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \psi_{y\nu} \mathbb{E}_t \nu_{t+1} \\ \psi_{\pi\nu} \mathbb{E}_t \nu_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & \sigma^{-1} & 0 \\ -\kappa & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{i}_t \\ \hat{\pi}_t \end{bmatrix}$$

Combine this with the backward looking part and thus:

$$\begin{bmatrix} 0 & 1 & -\phi_\pi \\ 1 & \sigma^{-1} & 0 \\ -\kappa & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{i}_t \\ \hat{\pi}_t \end{bmatrix} = \begin{bmatrix} 0 \\ (\psi_{y\nu} + \sigma^{-1} \psi_{\pi\nu}) \mathbb{E}_t \nu_{t+1} \\ \beta \psi_{\pi\nu} \mathbb{E}_t \nu_{t+1} \end{bmatrix} + \begin{bmatrix} \nu_t \\ 0 \\ 0 \end{bmatrix}$$

We can solve this system by inverting a matrix. The solution is:

$$\begin{bmatrix} \hat{y}_t \\ \hat{i}_t \\ \hat{\pi}_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\phi_\pi \\ 1 & \sigma^{-1} & 0 \\ -\kappa & 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ (\psi_{y\nu} + \sigma^{-1} \psi_{\pi\nu}) \mathbb{E}_t \nu_{t+1} \\ \beta \psi_{\pi\nu} \mathbb{E}_t \nu_{t+1} \end{bmatrix} + \begin{bmatrix} \nu_t \\ 0 \\ 0 \end{bmatrix} \right)$$

We can iterate backwards until the first period.

4 Empirical Analysis

In this section, I compare two methods of modeling LSAPs and two approaches to modeling ZIRP in DSGE models. Since Chen et al. (2012) study the market segmentation model carefully, I will only briefly show results. Here, I estimate the bonds-in-utility DSGE model that either incorporates a regime-switching monetary policy as 3.1 or a typical Taylor rule as 2.11. I extract the filtered states of those estimated DSGE models, and then, starting from the third quarter of 2010, I simulate the U.S. economy forward under four scenarios: no intervention and no shocks, only LSAPs intervention, only ZIRP for an extended period, and the combination of the LSAPs and the ZIRP for an extended period. I compare the pre-

dicted path of macro variables generated from the different models. When I evaluate ZIRP in the DSGE model with the regular Taylor rule, the PFRE method is used to simulate the economy. I will only explicate the estimation strategy of the regime-switching DSGE model. The description of the estimation procedure of the other non regime-switching model was omitted here. The Bayesian estimation methods for a linearized DSGE model with constant coefficients can be found, for example by An and Schorfheide (2007). Bayesian estimation combines prior information on the parameters with the likelihood function of the model to form the posterior distribution. In the regime-switching model, the optimal filter is no longer the Kalman Filter. I will first illustrate the optimal filter and the likelihood function for this regime-switching model, and then describe data, show estimation results, and make comparisons of simulation results.

4.1 *Optimal Filter and Likelihood Function*

Regime-switching model is complicated because usually we have to keep track of the long history of the distribution of the states, and the number of the states grows exponentially¹⁶. Fortunately, in my application, the distribution of the states at each time is degenerated, because I observe the interest rates, and thus deduce whether or not the economy is at the ZIRP regime in that period.

In this New Keynesian economy, the states are denoted by \mathbf{S}_t and the observables are denoted by \mathbf{y}_t . Let K_t denote the Markov regime-switching states and λ_t denote the probability at the ZIRP regime $K_t = 2$ at time t , thus $K_t = 1$, the normal regime, has probability $1 - \lambda_t$. Let \hat{R}_t denote the log deviation of the regime-switching interest rates from their ergodic mean. Its density function can be written as:

¹⁶Even with a 2-state Markov regime switching process, at time t , the number of states is 2^t .

$$P\left(\hat{R}_t\right)=\lambda_t^{\mathbf{1}_{\{R_t=0\}}}\left((1-\lambda_t) f_t\left(\hat{R}_t\right)\right)^{\mathbf{1}_{\{R_t>0\}}},$$

where $f_t\left(\hat{R}_t\right)$ is the conditional density, conditional on at the normal state. That is

$$P\left(\hat{R}_t|R_t>0\right)=f_t\left(\hat{R}_t\right).$$

Define the Dirac function as

$$\delta_{\tilde{x}}(x)=\begin{cases} 0 & \text{if } x \neq \tilde{x} \\ \infty & \text{if } x = \tilde{x} \end{cases} \quad \text{and} \quad \int \delta_{\tilde{x}}(x) dx = 1.$$

Using the Dirac function, I can express the density of the interest rates as

$$P\left(\hat{R}_t\right)=\lambda_t \delta_{\tilde{x}}(x)+(1-\lambda_t) f_t\left(\hat{R}_t\right).$$

The transition equations are

$$\mathbf{S}_t\left(K_t\right)=C\left(K_t\right)+G_t\left(K_t\right) \mathbf{S}_{t-1}\left(K_{t-1}\right)+R_t\left(K_t\right) \boldsymbol{\varepsilon}_t.$$

where all the coefficients are regime-dependent and the measurement equations are (no measurement error):

$$\mathbf{y}_t\left(K_t\right)=T \mathbf{S}_t\left(K_t\right).$$

Let $\bar{\lambda}$ denote the ergodic probability of the Markov chain and Σ_k denote the state-dependent variance-covariance matrix of the structural shocks:

$$\Sigma_k=E\left[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t'|K_t=k\right].$$

The algorithm of the optimal filter is as follows:

- Initializing at time $t = 1$, the mean of the states:

$$\bar{\mathbf{S}}_1 = \bar{\lambda}_1 (I - G(K_t = 1))^{-1} C(K_t = 1) + (1 - \bar{\lambda}_1) (I - G(K_t = 2))^{-1} C(K_t = 2),$$

and the variance,

$$\bar{P}_1 = \bar{\lambda}_1 X_1 + (1 - \bar{\lambda}_1) X_2,$$

where X_1 and X_2 solve the discrete Lyapunov matrix equations:

$$G(K_t = 1) X_1 G(K_t = 1)' - X_1 + R(K_t = 1) \Sigma_1 R(K_t = 1) = 0$$

and

$$G(K_t = 2) X_2 G(K_t = 2)' - X_2 + R(K_t = 2) \Sigma_2 R(K_t = 2) = 0$$

respectively.

- Forecasting $t + 1$ given t

– Transition equation

$$\begin{aligned}
& P(S_{t+1}, K_{t+1} | Y^t, \theta) \\
&= \int P(S_{t+1}, K_{t+1} | S_t, K_t) P(S_t, K_t | Y^t, \theta) d(S_t, K_t) \\
&= \int P(S_{t+1, -\hat{R}_{t+1}}, K_{t+1} | \hat{R}_{t+1}, S_t, K_t) P(\hat{R}_{t+1}, K_{t+1} | S_t, K_t) P(S_t, K_t | Y^t, \theta) d(S_t, K_t) \\
&= \int P(S_{t+1, -\hat{R}_{t+1}} | K_{t+1}, S_t, K_t) P(\hat{R}_{t+1} | K_{t+1}, S_t, K_t) P(K_{t+1} | S_t, K_t) P(S_t, K_t | Y^t, \theta) d(S_t, K_t) \\
&= \int P(S_{t+1, -\hat{R}_{t+1}} | K_{t+1} = 2, S_t, K_t) \delta_0(\hat{R}_{t+1} = 0) P(K_{t+1} = 2 | S_t, K_t) P(S_t, K_t | Y^t, \theta) d(S_t, K_t) \\
&\quad + \int P(S_{t+1} | K_{t+1} = 1, S_t, K_t) P(K_{t+1} = 1 | S_t, K_t) P(S_t, K_t | Y^t, \theta) d(S_t, K_t),
\end{aligned}$$

where $S_{t+1, -\hat{R}_{t+1}}$ denotes all the states excluding the interest rates. Since the density of the regime K_{t+1} , conditional on the last period states and regime, $P(K_{t+1} | S_t, K_t)$, is discrete, I can break the integral into two parts when it is in a ZIRP regime, and when it is in the normal regime. Notice that when it is in the ZIRP regime, I do not need to track the distribution of interest rates, because it is degenerated.

- – Measurement equation \implies likelihood function

$$\begin{aligned}
& P(y_{t+1} | Y^t, \theta) \\
&= \int P(y_{t+1} | S_{t+1}, K_{t+1}, Y^t, \theta) P(S_{t+1} | K_{t+1}, Y^t, \theta) P(K_{t+1} | Y^t, \theta) dS_{t+1} dK_{t+1} \\
&= P(K_{t+1} = 1 | Y^t, \theta) \int P(y_{t+1} | S_{t+1}, K_{t+1}, Y^t, \theta) P(S_{t+1} | K_{t+1}, Y^t, \theta) dS_{t+1} \\
&\quad + P(K_{t+1} = 2 | Y^t, \theta) \int P(y_{t+1, -\hat{R}_{t+1}} | S_{t+1}, K_{t+1}, Y^t, \theta) P(S_{t+1} | K_{t+1}, Y^t, \theta) dS_{t+1}.
\end{aligned}$$

- Updating

– Updating states

$$\begin{aligned}
& P(S_{t+1}, K_{t+1} | Y^{t+1}, \theta) \\
& \propto P(y_{t+1} | S_{t+1}, K_{t+1}, Y^t, \theta) P(S_{t+1}, K_{t+1} | Y^t, \theta) \\
& \propto P(y_{t+1} | S_{t+1}, K_{t+1}, Y^t, \theta) P(S_{t+1} | K_{t+1}, Y^t, \theta) P(K_{t+1} | Y^t, \theta) \\
& \propto P(y_{t+1} | S_{t+1}, K_{t+1}, Y^t, \theta) P(S_{t+1} | K_{t+1}, Y^t, \theta) P(K_{t+1} = 1 | Y^t, \theta) \\
& \quad + P(y_{t+1} - \hat{R}_{t+1} | S_{t+1} - \hat{R}_{t+1}, K_{t+1}, Y^t, \theta) P(S_{t+1} - \hat{R}_{t+1} | K_{t+1}, Y^t, \theta) P(K_{t+1} = 2 | Y^t, \theta).
\end{aligned}$$

- – Updating states probability

Since I observe the data y_{t+1} , I observe the interest rate. If $R_{t+1} = 0$, I deduce that

$$P(K_{t+1} = 1 | Y^{t+1}) = 0, \text{ and } P(K_{t+1} = 2 | Y^{t+1}) = 1$$

and *vice versa*. So I do not need to track the long history of the states, because when I know the history of Y^t , I know the history of the states for sure. The distribution of the states at each time is degenerated. In practice, any quarterly Federal Funds rate that is smaller than 40bp is treated as zero interest rate.

4.2 Data

I use the same observables as Chen et al. (2012). I use the United States quarterly data from the third quarter of 1987 (1987q3) to the second quarter of 2010 (2010q2) for the following seven series: real GDP per capita, hours worked, real wages, core personal consumption expenditures (PCE) deflator, nominal effective federal funds rate, the 10-year Treasury constant maturity yield, and the ratio between long-term and short-term U.S.

Treasury debt.¹⁷ I use long-term bond yields because I want to match the term structure implied by the model with that of the data. Also bond data are used to identify a bond supply shock. All data are extracted from the Federal Reserve Economic Data (FRED) maintained by the Federal Reserve Bank of St. Louis. The mapping between these observable variables and the state variables in the DSGE models is

$$\begin{aligned}
\Delta Y_t^{obs} &= 100(\gamma + \hat{Y}_{z,t} - \hat{Y}_{z,t-1} + \hat{z}_t), \\
L_t^{obs} &= 100 \left(L + \hat{L}_t \right), \\
\Delta w_t^{obs} &= 100(\gamma + \hat{w}_{z,t} - \hat{w}_{z,t-1} + \hat{z}_t), \\
\pi_t^{obs} &= 100(\pi + \hat{\pi}_t), \\
r_t^{obs} &= 100(r + \hat{r}_t), \\
r_{L,t}^{obs} &= 100(r_L + \hat{r}_{L,t}), \\
B_t^{ratio,obs} &= \frac{P_{L,t} B_t^L / (P_t Z_t)}{B_t / (P_t Z_t)},
\end{aligned}$$

where all state variables are in deviations from their ergodic steady state values (corresponding to the ergodic steady state R for the policy rate), $\pi \equiv \ln(\Pi)$, $r \equiv \ln(R)$, and $r_L \equiv \ln(R_L)$.

I construct the real GDP per capita series by dividing the nominal GDP series by the population and the GDP deflator. The observable ΔY_t^{obs} , the growth rate of real GDP, corresponds to the first difference in logs of this series, multiplied by 100. I measure the labor input by the log of hours of all persons in the non-farm business sector divided by the population. Real wages correspond to the nominal compensation per hour in the non-farm business sector, divided by the GDP deflator. Δw_t^{obs} , the growth rate of real wage, is the first difference in logs of this series, multiplied by 100. The log-difference of the quarterly

¹⁷I use an extended sample, starting in 1975q1, to initialize the filter, but the likelihood function itself is evaluated only for the period starting in 1987q3, conditional on the previous sample.

personal consumption expenditures (PCE) core price index is the measure of inflation. I use the effective federal funds rate as the measure of the nominal short-term rates and the 10-year Treasury constant maturity rates as the measure of the nominal long-term interest rates. Since in the model I do not differentiate between the government and the central bank, short-term bonds include both government bonds with maturity shorter than one year and the central bank liabilities in the form of reserves, vault cash, and deposits and currency. Long-term bonds include all the government bonds with maturity longer than one year, consistent with the LSAPs II announcement.

4.3 *Prior Choice*

Tables 1 and 2 (columns two to four) summarize the prior distributions of each parameter in the regime-switching DSGE model. I fix the coefficient of relative risk aversion σ at 2, and the steady state of the ratio of long-term bonds to short-term bonds at 1.01, which is consistent with the average of this series in the data. I use Gamma distributions for the prior distributions of the parameters that economic theory suggests must be positive. For those parameters that are defined over the interval $[0, 1]$, I use the Beta distribution. For the standard deviation of the structural shocks, I use the Inverse-Gamma distribution.

The ergodic mean for inflation is centered at 2%, consistent with the Federal Open Market Committee’s long-term inflation mandate. The steady state annualized growth rate of output is centred at 2.5%. The prior distribution of the discount factor implies the mean of the annualized real interest rate is 2%. The spread between the short-term rates and long-term rates has a mean of 0.75% (annualized) at its prior distribution.

I follow Del Negro and Schorfheide (2008) to choose the priors for the standard parameters in the DSGE models. As in Chen et al. (2012), the dividend payment parameter k for the long-term bonds is calibrated to imply a duration of 30 quarters, which is consistent with the average duration of the U.S. 10-year Treasury bonds in the secondary market.

Table 1 contains three non-standard parameters ($\tilde{\nu}$, P_{11} , and P_{22}) specific to this regime-switching bonds-in-utility model, which controls the elasticity of households' portfolio mix in response to the long-term rate, the Markov switching probability of staying in the normal regime at time $t + 1$ when it is in the normal regime at time t , and the Markov switching probability of staying in the ZIRP regime at time $t + 1$ when it is in the ZIRP regime at time t . $\tilde{\nu}$ is centered at 0.1 at the prior. Harrison (2012) uses a parameter with a similar role, and he calibrates this parameter to be 0.09. Andrés et al. (2004) estimate a similar parameter to be 0.045, which describes the elasticity of the risk premium to a change in the ratio of long-term bonds to money. I do not have money in my model, but the short-term bonds fill a similar role as money because it is more liquid than long-term bonds. Bernanke, Reinhart, and Sack (2004) suggest that a 10% reduction in the stock of long-term bonds associated with the U.S. Treasury buy-backs reduces long yields by around 100 basis points. The second round large-scale asset purchases is equivalent to a 25% reduction in long-term bonds¹⁸. This suggests a value for $\tilde{\nu}$ around 0.25. My prior mean lies in between those estimates. P_{11} is centered at 0.99, which implies an expected duration of staying in the normal regime is 25 years. P_{22} is centered at 0.85 at prior, which implies an expected duration of staying in the ZIRP regime is 6.7 quarters, consistent with what is observed in the data.

The prior for the price rigidity parameter, ζ_p , is centred at 0.5 with a standard deviation of 0.1, as in Smets and Wouters (2007). The interest rate smoothing parameter, ρ_r , is centered at 0.7. The interest rate feedback to output growth, ϕ_y , is centred at 0.4, and the feedback to inflation, ϕ_π , is centred at 1.5 at priors.

All the structural shocks follow AR(1) processes. Their autocorrelation coefficients are centred at 0.75 or 0.8, with the exception of productivity shocks whose autocorrelation coefficient is centered at 0.4, because this process characterizes the transitory shock to the growth rate of the technology process.

¹⁸It corresponds to roughly a 24% reduction in the ratio of long-term bonds to short-term bonds.

4.4 *Parameter Posterior Distribution*

In order to obtain the posterior distribution of the parameters, I first obtain the posterior mode by maximizing the likelihood function. The last column of tables 1 and 2 report the posterior mode of each parameter. I then use the random walk Metropolis Hastings algorithm to draw from the posterior distributions. I store those parameter draws and use them for simulation exercises discussed later.

The Markov switching probabilities are well identified because, although the priors are concentrated at their mean, the posterior modes of the transition probabilities are very distinguishable from the prior means. The posterior distributions indicate that the expected duration of staying in the normal regime is 24.15 quarters, and the expected duration of staying in the ZIRP regime is 4.5 quarters. One may argue that data seem to suggest that we have been in the ZIRP regime for at least 14 quarters (from 2009Q1 to 2012Q2). There are two reasons why the estimated duration is substantially shorter than this period. First, the data in my estimation stops at the second quarter of 2010, by which there were only 6 quarters of zero interest rate policy. Second, I treat the 8 quarters from 2002Q4 to 2004Q3 as a ZIRP regime (quarterly FFR is less than 40 basis points) so that we have observations of exiting the ZIRP regime. The time of staying in the ZIRP regime is also short here.

4.5 *The Efficacy of the LSAPs in DSGE models*

Having estimated the DSGE models, I abstract the filtered states, and, starting from 2010Q3, I simulate U.S. economy forward for 20 quarters under two scenarios. Under the first scenario, there is no intervention from the central bank, and all the structural shocks are zero. So, output should gradually go back to its long-term trend, and inflation and interest rates should gradually go back to their steady states. Under the second scenario, the economy is under the intervention of asset purchases by the central bank simulated to

mimic the Federal Reserve’s second round LSAPs, a \$600 billion reduction of long-term debt in the hands of the private sector. The central bank buys long-term bonds (in exchange for the short-term bonds) over the course of the first four quarters, holds the ratio of the market value of the long-term bonds to that of the short-term bonds constant for the next two years, and gradually reverts the LSAPs program over the final two years. Figure 1 illustrates the path of the ratio of the market value of long-term bonds to that of the short-term bonds in the hands of the private sector following the LSAPs by the central bank. In the regime-switching bonds-in-utility model, this simulation is achieved by feeding the unanticipated shocks to the bond supply rule 2.13. In the non-regime-switching bonds-in-utility model, with a regular Taylor rule, agents have perfect knowledge of the bond purchases path, and the equilibrium is solved by the PFRE solution method explained in section 3.2.

I simulate the LSAPs 500 times using the parameter draws from the posterior distributions and take the average of the predicted path. Figure 2 shows the predicted path generated by the non-regime-switching bonds-in-utility model, and Figure 3 shows the predicted path generated by the regime-switching bonds-in-utility model¹⁹. The red lines in those two figures are the predicted path without intervention, the blue lines are the predicted path under the LSAPs, and the black dots are actual observations. Output is per capita level data, while the units of the other variables are percentage measured quarterly. It is clear from those figures that the effects of the LSAPs are unlikely to be significant no matter what model we use, and whether or not agents are taken by surprise. At each time point, I take the percentage difference of the macro variables between the path with and the path without the LSAPs intervention, and sum up the difference over the 20 quarters to measure the total

¹⁹ Another complication in the simulation in the regime-switching DSGE model is that agents have uncertainty over the future states. There are 2^t possible states at time t . To maintain tractability, I collide the states with similar history and only keep track of 16 states at each period (See Schorfheide (2005) for how this can be achieved.). The predicted path of the macro variables plotted is thus the probability weighted average of those 16 states.

effects. The non-regime-switching bonds-in-utility DSGE model suggests on average²⁰ the LSAPs increase output level by 0.34% and inflation by 0.16% over the course of 20 quarters. The regime-switching model suggests a slightly bigger effect, on average the LSAPs increase output level by 1.03% and inflation by 0.25% over the course of 20 quarters. This finding agrees with the results reported by Chen et al. (2012). Section 5 investigate further why the effects of the LSAPs are so small measured in the DSGE models and evaluate their effects with VARs.

4.6 *The Efficacy of the ZIRP in DSGE models*

Zero interest rate policy is effective in boosting output and inflation. Both of the models considered suggest substantial effects of the ZIRP. When I simulate the U.S. economy under the ZIRP for an extended period, I consider keeping interest rates at zero for four quarters at the regime-switching model and keeping interest rates at the 2010Q2 level for four quarters in the model where the ZIRP is implemented by the PFRE. In the regime-switching model, at each period, agents *ex ante* always attach certain probability of exiting the ZIRP regime in the next period, and the ZIRP regime is realized for four quarters *ex post*. In the PFRE model, agents know that the ZIRP will be kept for four quarters. I choose four quarters because although the Federal Reserve announced on September 13th, 2012 that the ZIRP will last to "at least mid-2015", participants of the Blue Chip Survey, professionals and economists, expected the ZIRP to last four or five quarters at the end of 2010 when the LSAPs II were implemented. Figure 4 and Figure 5 show the predicted path under the ZIRP generated by the PFRE model and the regime-switching model. The red lines in those two figures are the predicted path without the ZIRP, the blue lines are the predicted path with the ZIRP, and the black dots are actual observations. The regime-switching bonds-

²⁰"On average" means average over parameter uncertainty.

in-utility DSGE model suggests on average the ZIRP increases output level by 12.83% and inflation by 2.08% over the course of 20 quarters. The non-regime-switching model where the ZIRP is implemented by the PFRE suggests a two fold stronger effect on output level and five fold stronger stimulus to inflation: On average the ZIRP increases output level by 25.01% and inflation by 11.71% over the course of 20 quarters. As mentioned earlier, those two models are fundamentally different in how agents formulate expectations about the future monetary policy. The central bank's "extended period" language is treated as completely credible by the agents in the PFRE model, while in the regime-switching model, agents ignore the central bank's forward guidance. Figure 6 compares the predicted path of inflation generated by those two models. The red line is the predicted path from the regime-switching model and the green line is the predicted path from the PFRE model. The black dots are actual data. It demonstrates that actual path is a lot closer to the path from the regime-switching model.

Figure 14 summarizes the effects of the LSAPs and the ZIRP in the DSGE models. At each time of the simulated path, I take the percentage difference of the macro variables with and without intervention, and sum up over 20 quarters. This figure plots the total effects. The color green represents the bonds-in-utility model. The squares are mean responses and the circles reflects the parameter uncertainty. The blue square reports the mean effects measured in the market segmentation model reported by Chen et al. (2012). This figure clearly shows that the effects of LSAPs are very small, while the efficacy of ZIRP is substantial, and crucially depends on the models.

4.7 *The Efficacy of the Combination of the LSAPs and the ZIRP*

Since the effects of the LSAPs alone is very small, unsurprisingly, the effects of the combination of the LSAPs and the ZIRP are dominated by the effects of the ZIRP. Figure

7 (the PFRE model) and Figure 8 (the regime-switching model) shows that the predictive paths of the macro variables under the ZIRP (blue lines) and under the combination of the LSAPs and the ZIRP (green lines) are almost indistinguishable from each other²¹. Chen et al. (2012) also emphasize the importance of the Federal Reserve's commitment to keep the interest rates at zero for an extended period.

5 The Efficacy of the LSAPs in VAR models

DSGE models impose strict cross equation restrictions. The DSGE models considered in this work impose a strong assumption on how LSAPs are identified: Equation 2.13 shows that the bond supply follows an AR(1) process exogenously, and other structural shocks do not affect the dynamics of bonds. LSAPs were never implemented before in the U.S. history until the recent recession; however, DSGE models use the covariance relationship between the bonds and other macro variables in the historical data to "identify" the effects of the assets purchases to macro variables. In the data, the variation of the bonds in the past could be due to an entirely different reason. It could be a demand shock. For example, by preferred habitat theory, long-term interest rates could experience a large and long-lasting drop because of a demand shock of a long-maturity clientele such as pension fund, which in turn would stimulate private borrowing and investment. This implies a positive covariance between long-term bond quantity in the hands of private sector and macrovariables: opposite of the covariance relationship the LSAPs assume. Although by construction the LSAPs should have a positive effect in DSGE models, the insignificant effects found in the DSGE models are probably due to the identification strategy of those models: the covariances between bonds and macro variables in the past are not informative about the effectiveness of the LSAPs. To further investigate how much of the finding that the effects of the LSAPs are

²¹Red lines are the predictive path under no intervention and no shocks.

small is due to the strict restrictions imposed by the DSGE models, I compare the DSGE models with the VARs. I ask the question, what are the effects of the LSAPs in an estimated VAR using the identification restrictions imposed by the DSGE models? What happens if I further relax those restrictions?

5.1 VAR with Exogenous Restrictions

The assumption of the DSGE models that the bond supply follows an AR(1) process exogenously, and other structural shocks do not affect the dynamics of bonds provides an exogenous restriction to identify a bond supply shock in a VAR model. I estimate the following VAR:

$$\mathbf{y}_{1,t} = \Phi_0 + \Phi_1 \mathbf{y}_{t-1} + \Phi_2 \mathbf{y}_{t-2} + \Phi_3 \mathbf{y}_{t-3} + \Phi_4 \mathbf{y}_{t-4} + \Psi (y_{2,t} - C - \rho_B y_{2,t-1}) + \mathbf{u}_{1,t}$$

$$y_{2,t} = C + \rho_B y_{2,t-1} + \sigma_B \varepsilon_{B,t}$$

where $\mathbf{y}_{1,t}$ are the growth rate of output, inflation, long rates, and short rates, and $y_{2,t}$ is the ratio of the market value of the long bonds to that of the short bonds. The definitions of those variables are described in section 4.2. $\mathbf{u}_{1,t}$ are measurement errors. $\varepsilon_{B,t}$ is the bond supply shock. $\mathbf{y}_{1,t}$ are affected by the bond supply shock, but the bond supply is exogenous and unaffected by other macro variables. To simulate the Federal Reserve's second round LSAPs, I calibrate the bond shocks as described in section 4.5. In order to assess the effects of ZIRP, I also identify a monetary policy structural shock and impose ZIRP by unanticipated monetary policy shocks. I identify this monetary policy shock by short-run restriction, that is, monetary authority shocks do not affect the private sector's activity on impact. Suppose

the first two elements of $\mathbf{y}_{1,t}$ are the growth rate of output and inflation. Let $\Sigma_{\mathbf{u}}$ denote the variance and covariance matrix of \mathbf{u}_1 , and let Σ_{tr} denote the Cholesky decomposition of $\Sigma_{\mathbf{u}}$. I draw a unit length vector q , the first two elements of which equal zero. $\Sigma_{tr} \cdot q$ identifies the impact of the monetary shock to the observables $\mathbf{y}_{1,t}$. Finally, I simulate the economy forward with the estimated VAR model. Figure 9 shows the predicted path under no intervention or shocks, under the LSAPs, and under the ZIRP for four quarters. The red line shows the predicted path of the macro variables under no intervention and no shocks, where output is the per capita output level, inflation is the quarterly percentage change of the core PCE, short rates are quarterly federal funds rate, and long rates are quarterly rates for the 10-year Treasury constant maturity bonds. The blue and green lines are the corresponding paths under the LSAPs and the ZIRP. A comparison between the red and the blue lines shows no evidence of a positive effect of the LSAPs, while ZIRP has a stimulative effect (difference between the green line and the red line). Figure 10 adds another grey line on each panel of the Figure 9. This grey line on each panel represents the predictive path of the corresponding macro variable under the intervention of the combination of the LSAPs and the ZIRP. Unlike the case in DSGE models, the combination effects seem to be dominated by the effects of LSAPs since the grey line is very close to the blue line which is the predictive path from the intervention of LSAPs only.

5.2 VAR with Sign Restrictions

The exogenous restriction is a very strong assumption. Whether or not it is valid is subject to debate. The DSGE model also implies certain directional restrictions of the responses of the macro variables to the LSAPs. The DSGE models imply that the LSAPs reduce long-term rates, stimulate output and inflation. Those directional restrictions provide

the sign restrictions to identify a risk premium shock of the following VAR.

$$\mathbf{y}_t = \Phi_0 + \Phi_1 \mathbf{y}_{t-1} + \Phi_2 \mathbf{y}_{t-2} + \Phi_3 \mathbf{y}_{t-3} + \Phi_4 \mathbf{y}_{t-4} + \mathbf{u}_t,$$

where \mathbf{y}_t is a collection of the growth rate of output, inflation, short rates, and long rates. I assume that the risk premium shock has zero impact on short-term rates, reduces the long-term bond rates, and increases output and inflation on impact²². I also calibrate the size of this shock so that the mean reduction of the long-term bond rates on impact is 30 basis point, which lies in the mid-range of the values reported by empirical studies of the effects of LSAPs. The monetary policy shock is identified by sign restrictions. The monetary policy shock increases short and long rates on impact, but decreases output growth rate and inflation on impact. This identification scheme is very similar to Baumeister and Benati (2010) and Chen, Cúrdia, and Ferrero (2011) working paper. Baumeister and Benati (2010) use zero and sign restrictions to identify a risk premium shock that decreases long rates by 1 percent, and Chen et al. (2011) calibrate whatever size of the bond supply shock necessary to decrease the long-term bond rates by 30 basis point on impact. Figure 11 shows the simulation results of the same experiment: I simulate the economy forward under no intervention and no shock, under the LSAPs, and under the ZIRP. The red line is the predicted path of the macro variables under no intervention, averaged over different parameter draws from the posterior distributions. The blue line is the predicted path of the macro variables under the LSAPs intervention, and the green line is the predictive path under the ZIRP. The ZIRP has a substantial effect as measured in the VAR model. There is potentially a positive effect of the LSAPs, but it is considerably uncertain²³. Figure 12 plots the estimate of the identified set of the effects of the LSAPs. The green line is the counterfactual scenario when there is no policy

²²The DSGE models suggest those sign restrictions. The empirical question is then, how big are the effects of the policies.

²³See Figure 15 where the uncertainty is reflected by the ellipse in red.

intervention, while the red line is the mean of the predicted path of the macro variables under the LSAPs II intervention. The blue lines plot the identified set. The effects of LSAPs could be potentially substantial, but it is considerably uncertain. Figure 13 adds another grey line on each panel of the Figure 11. This grey line on each panel represents the predictive path of the corresponding macro variable under the intervention of the combination of the LSAPs and the ZIRP. It is interesting to notice that in the VAR model with sign restrictions the effects of the combination of those two policies seem a weighted average of the LSAPs and the ZIRP. The effects of ZIRP to output dominates the effects of LSAPs, while the effects of the LSAPs to inflation dominated ZIRP. Figure 15 summarizes the effects of the LSAPs and the ZIRP aggregate over 20 quarters. I take the log-difference of the predicted macro variables with and without intervention at each time point and sum up over 20 quarters to reflect the total effects. The squares are the mean effects, and the circles reflects the uncertainty of the parameter draws. The pink color represents the results generated by the VAR with the exogenous restriction, while the red color represents the results generated by the VAR with the sign restrictions. One reason why the effects of the LSAPs and the ZIRP are considerably uncertain is the partial identification of the sign restrictions.

6 Conclusions

Given the unusual size and scope of the unconventional monetary policies, it is critical for economists to construct models capable of assessing their effectiveness and guiding policy. This paper develops a new approach to modeling the ZIRP, which not only fits the macro data featuring a persistent period of extremely low interest rates, and generates a predicted path closer to the actual path, but also provides a plausible mechanism for modeling the exit of the zero interest rate policy. Also, by cross-evaluation of the different models of the LSAPs and the ZIRP, I find that the Federal Reserve’s commitment to an extended period

of low interest rates is likely to be effective in boosting the economy while the efficacy of LSAPs is uncertain.

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Table 1: *Parameter Prior and Posterior Distribution: Structural Parameters.*

	Dist	Prior		Posterior
		Mean	Std	Mode
400γ	G	2.5	0.5	1.9263
400π	G	2.0	0.5	2.0711
$400(\beta^{-1} - 1)$	G	1.0	0.25	0.8210
400ζ	G	0.75	0.25	0.6633
ν	G	2.0	0.5	0.5682
ζ_p	B	0.5	0.1	0.9066
ρ_r	B	0.7	0.1	0.7958
ϕ_π	G	1.5	0.25	1.8069
ϕ_y	G	0.4	0.2	0.3261
P_{11}	B	0.99	0.05	0.9586
P_{22}	B	0.85	0.10	0.7793
$\tilde{\nu}$	G	0.10	0.05	0.0797

Table 2: Parameter Prior and Posterior Distribution: Shock Process Parameters.

	Dist	Prior		Posterior
		Mean	Std	Mode
ρ_z	B	0.4	0.2	0.1560
ρ_b	B	0.75	0.1	0.7624
ρ_ϕ	B	0.75	0.1	0.9694
ρ_B	B	0.8	0.1	0.9154
ρ_ζ	B	0.8	0.1	0.9307
ρ_g	B	0.75	0.1	0.9329
$100\sigma_z$	IG1	0.5	4	0.5542
$100\sigma_b$	IG1	0.5	4	1.5873
$100\sigma_\phi$	IG1	0.5	4	1.2965
$100\sigma_B$	IG1	0.5	4	0.1162
$100\sigma_m$	IG1	0.5	4	0.1646
$100\sigma_\zeta$	IG1	0.5	4	0.3201
$100\sigma_g$	IG1	0.5	4	4.2947

7 Figures

Fig. 1: Simulated path of the ratio of the market value of long term bonds to that of the short-term bonds

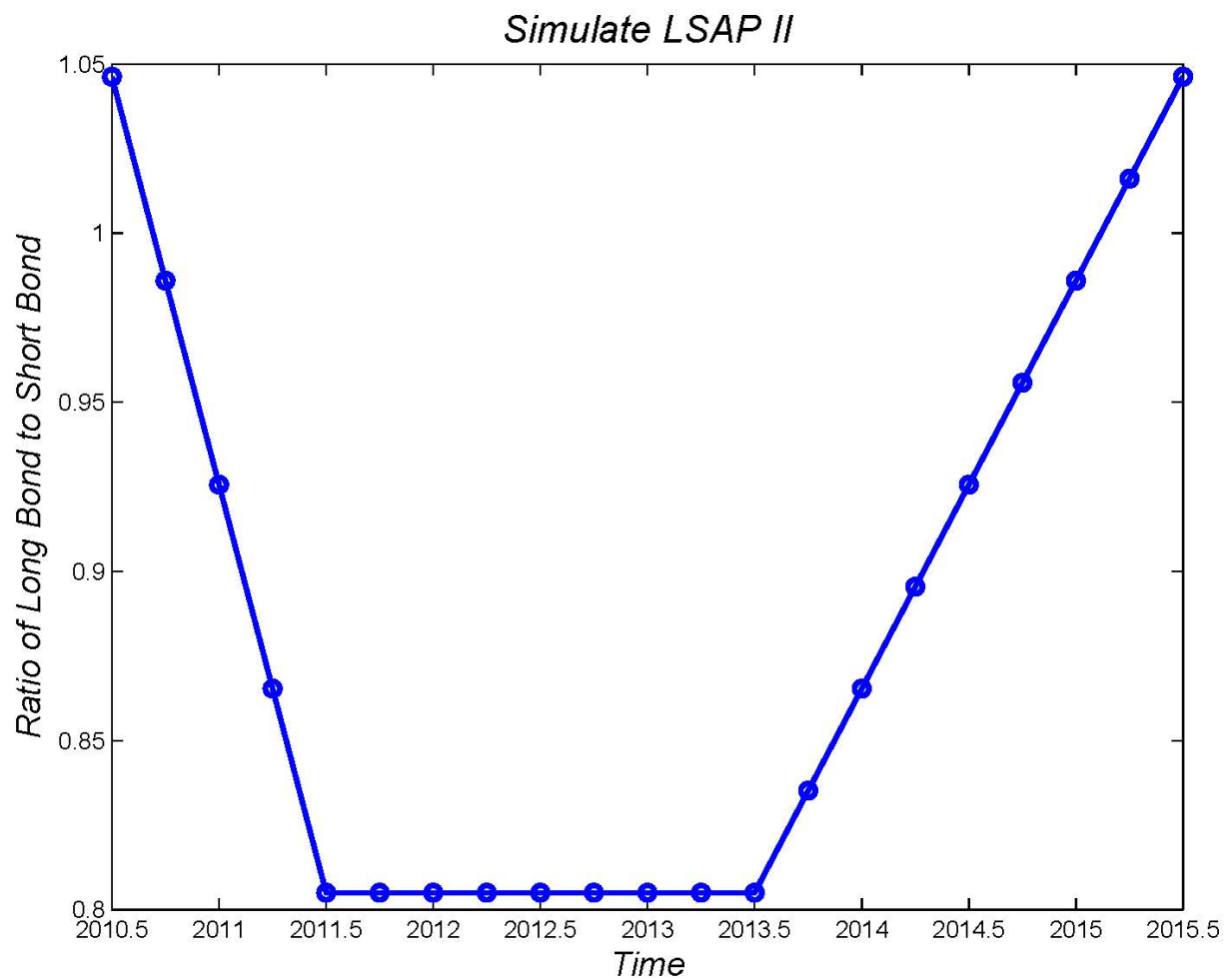


Fig. 2: Simulate the U.S. economy forward from 2010Q3 under the LSAPs II intervention in the NON-regime-switching bonds-in-utility DSGE model with standard Taylor rule. The red lines show the mean of predicted paths of the macro variables without shocks and under no intervention. The blue lines show the mean of predicted paths of the macro variables under the LSAPs II intervention generated by the same model

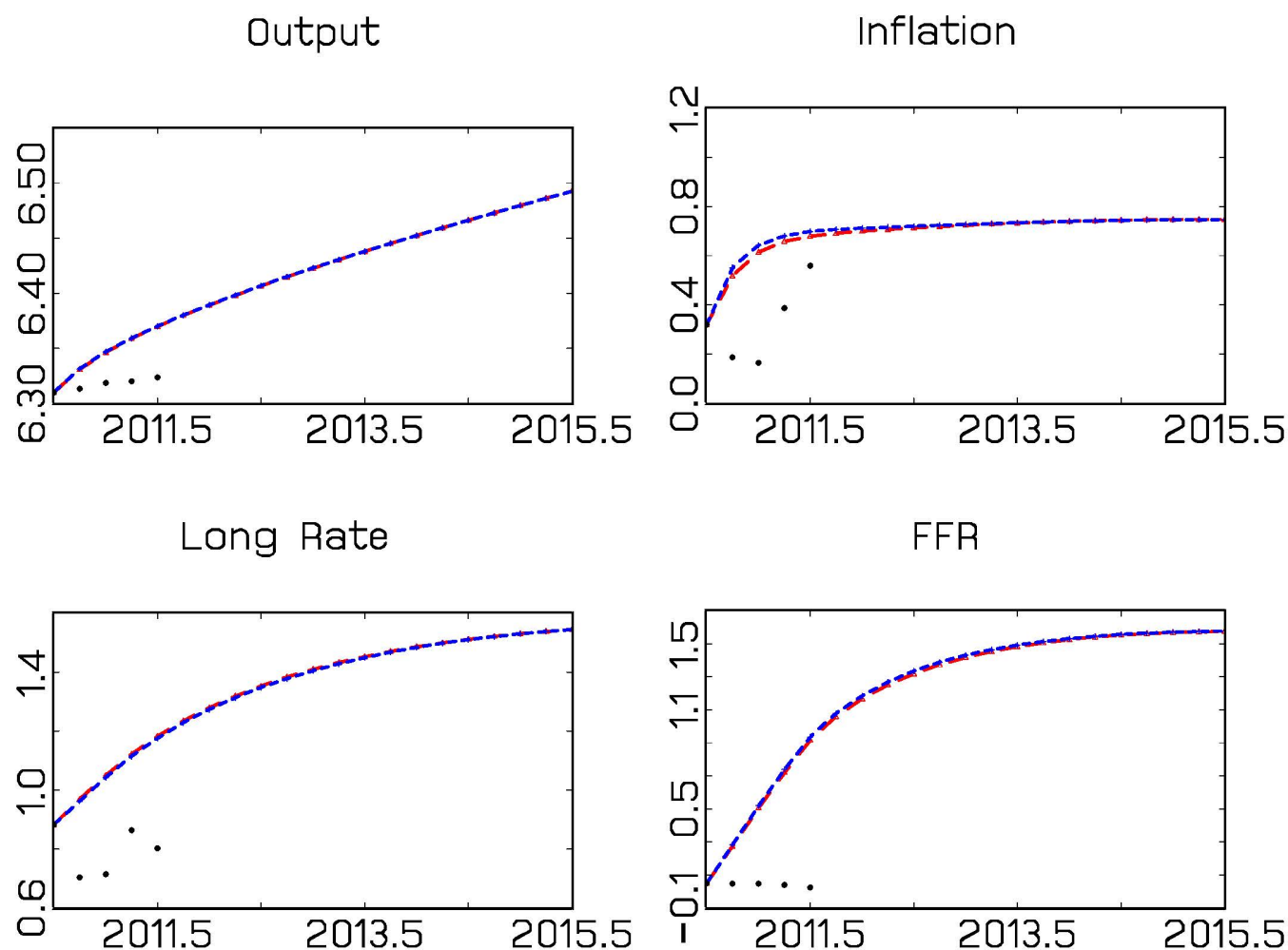


Fig. 3: Simulate the U.S. economy forward from 2010Q3 under the LSAPs II intervention in the regime switching Bonds-in-utility DSGE model. The red lines show the mean of predicted paths of the macro variables without shocks and under no intervention. The blue lines show the mean of predicted paths of the macro variables under the LSAPs II intervention generated by the same model

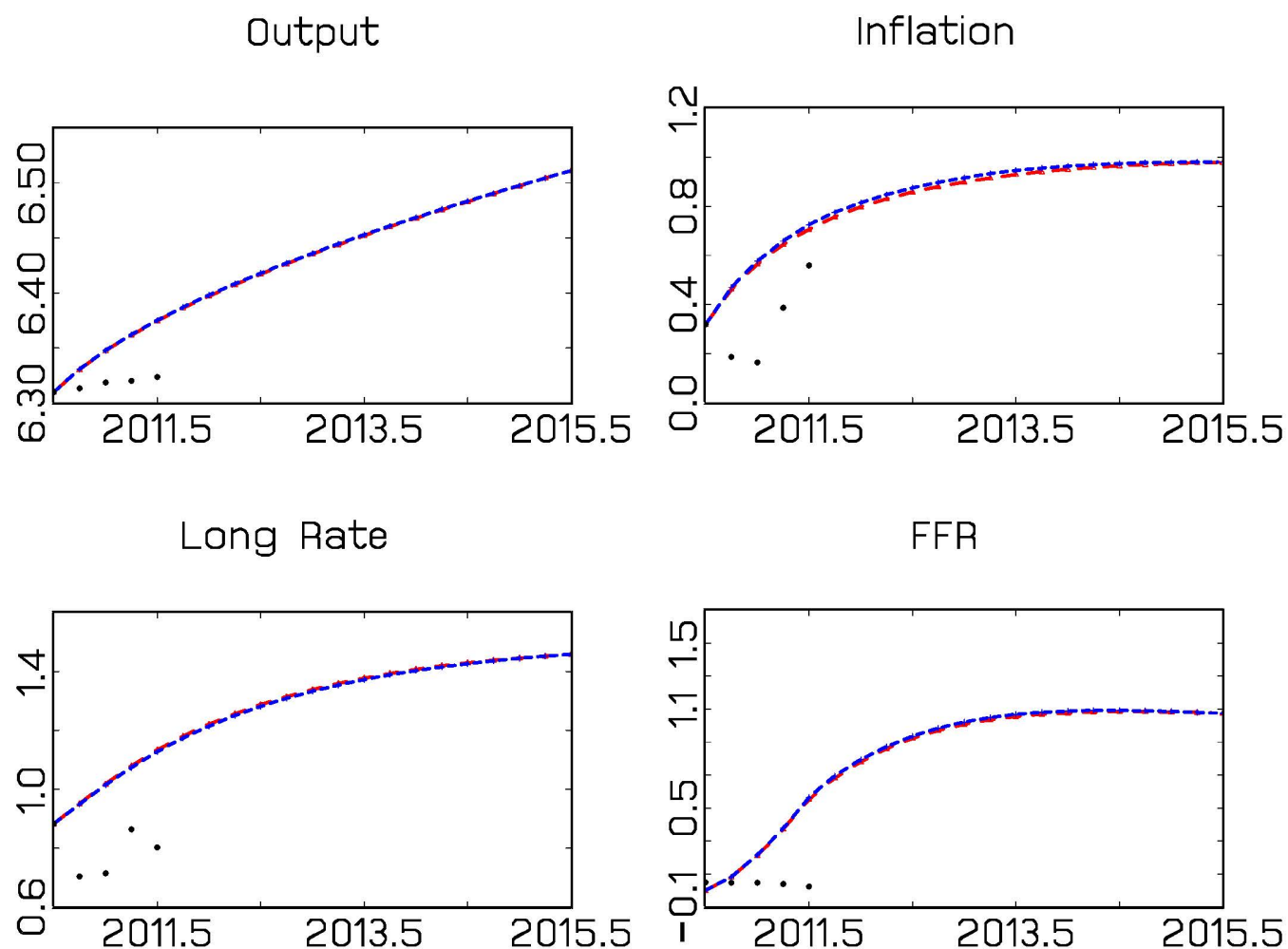


Fig. 4: Simulate the U.S. economy forward from 2010Q3 under the ZIRP intervention implemented by the PFRE. The red lines show the mean of predicted paths of the macro variables without shocks and under no intervention generated by the bonds-in-utility DSGE models. The blue lines show the mean of predicted paths of the macro variables under the ZIRP for four quarters generated by the same model

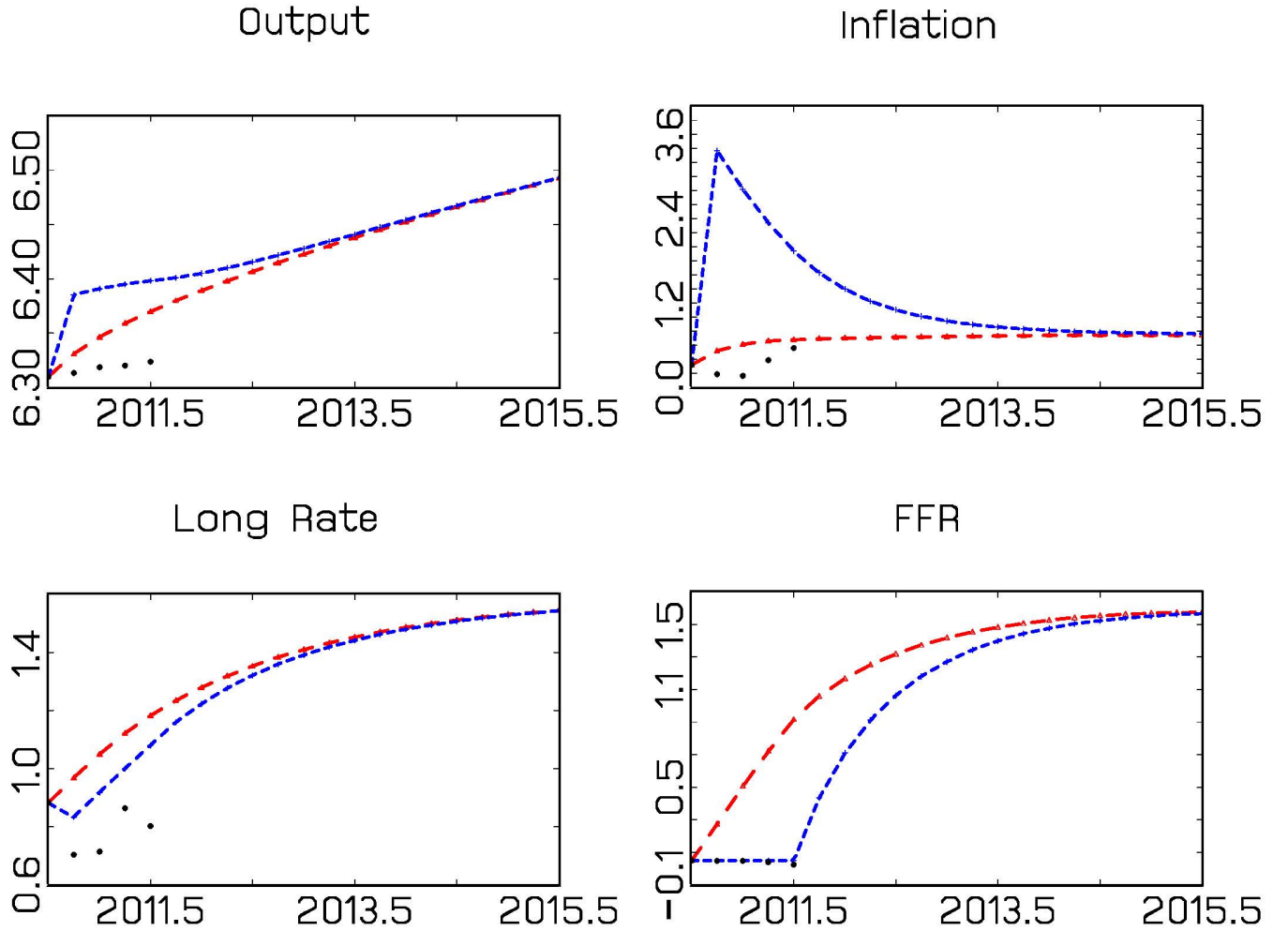


Fig. 5: Simulate the U.S. economy forward from 2010Q3 under the ZIRP intervention in the regime switching bonds-in-utility DSGE model. The red lines show the mean of predicted paths of the macro variables without shocks and under no intervention. The blue lines show the mean of predicted paths of the macro variables under the ZIRP for four quarters generated by the same model

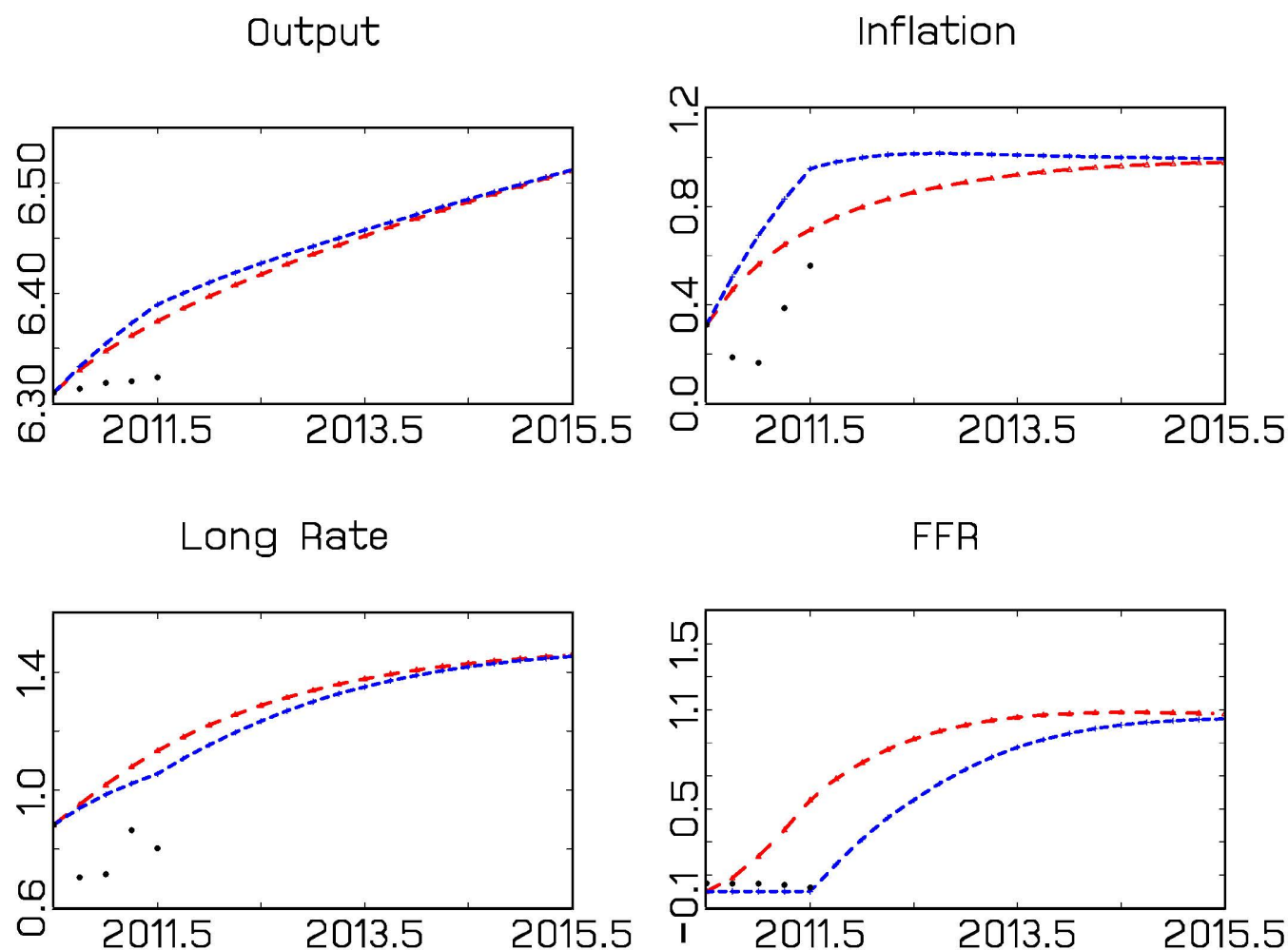


Fig. 6: Compare the predicted path of inflation generated by two different models of ZIRP. Red represents regime switching model while green stands for PFRE model.

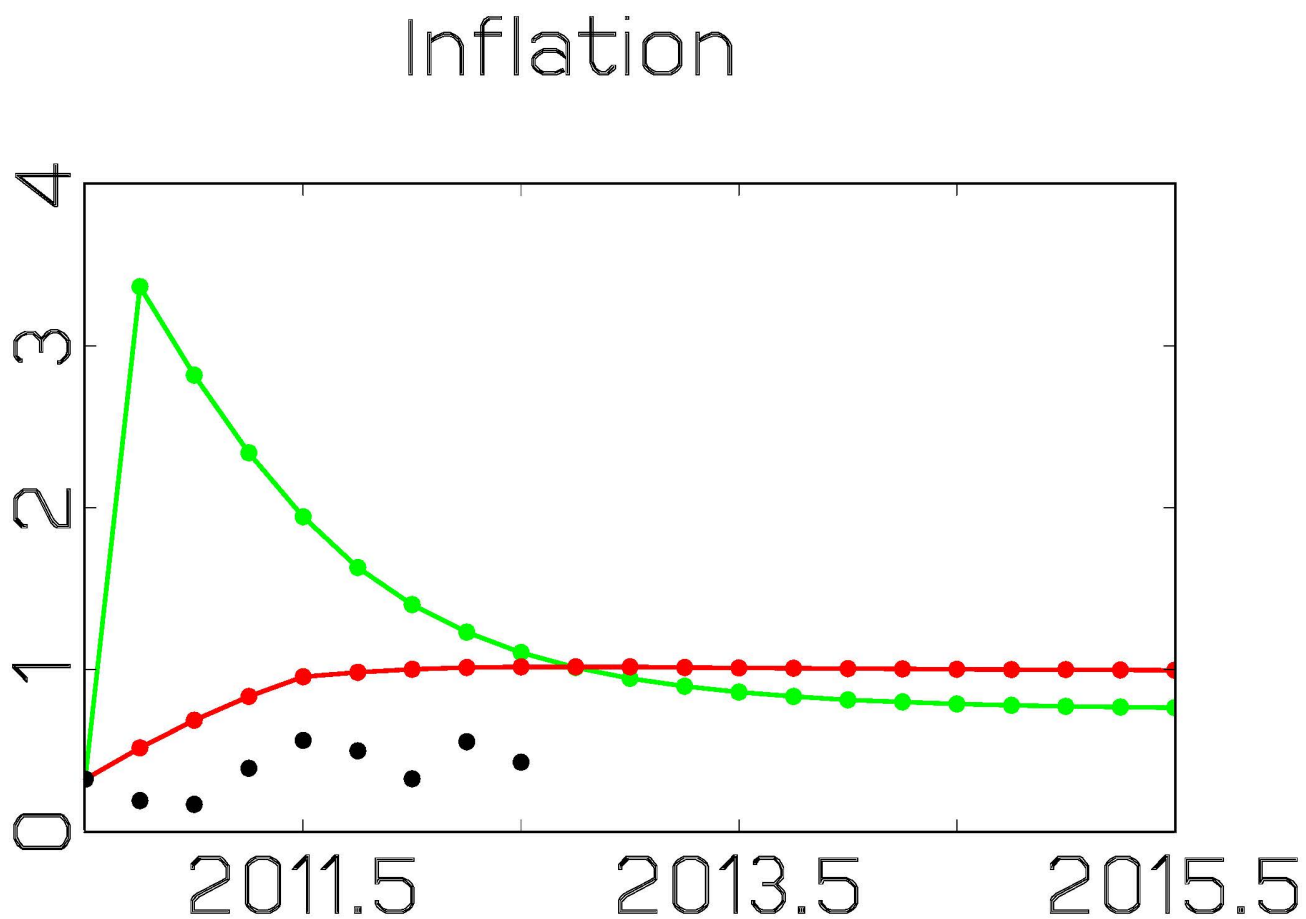


Fig. 7: Simulate the U.S. economy forward from 2010Q3 under the ZIRP intervention implemented by the PFRE and under the combination of the LSAPs and the ZIRP. The red lines show the mean of predicted paths of the macro variables without shocks and under no intervention generated by the bonds-in-utility DSGE models. The blue lines show the mean of predicted paths of the macro variables under the ZIRP for four quarters generated by the same model, and the green lines show the predictive paths under the intervention of the combination of the LSAPs and the ZIRP for four quarters

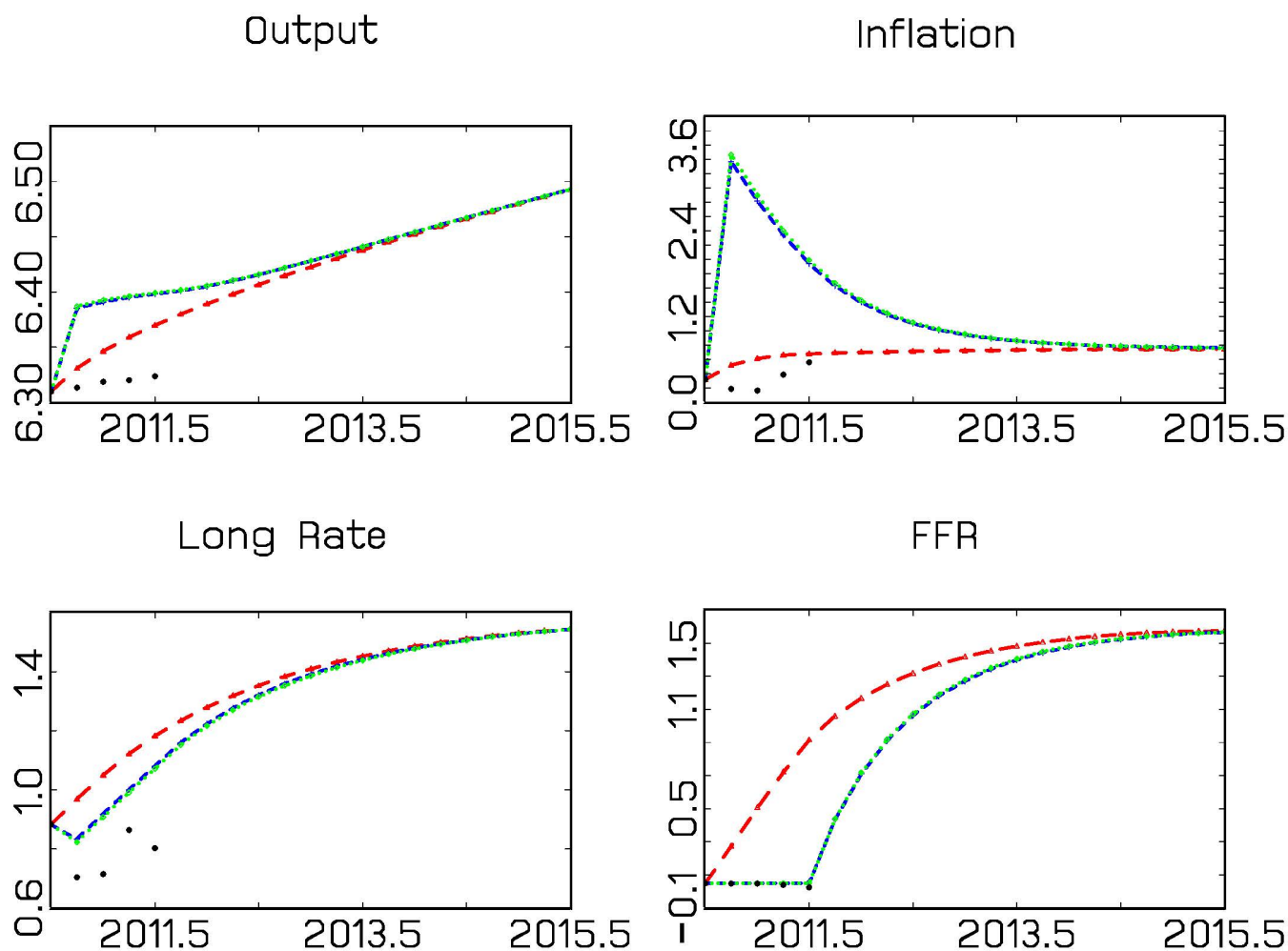


Fig. 8: Simulate the U.S. economy forward from 2010Q3 under the ZIRP intervention implemented by the regime-switching bonds-in-utility model and under the combination of the LSAPs and the ZIRP. The red lines show the mean of predicted paths of the macro variables without shocks and under no intervention, the blue lines show the mean of predicted paths of the macro variables under the ZIRP for four quarters, and the green lines show the predictive paths under the intervention of the combination of the LSAPs and the ZIRP for four quarters

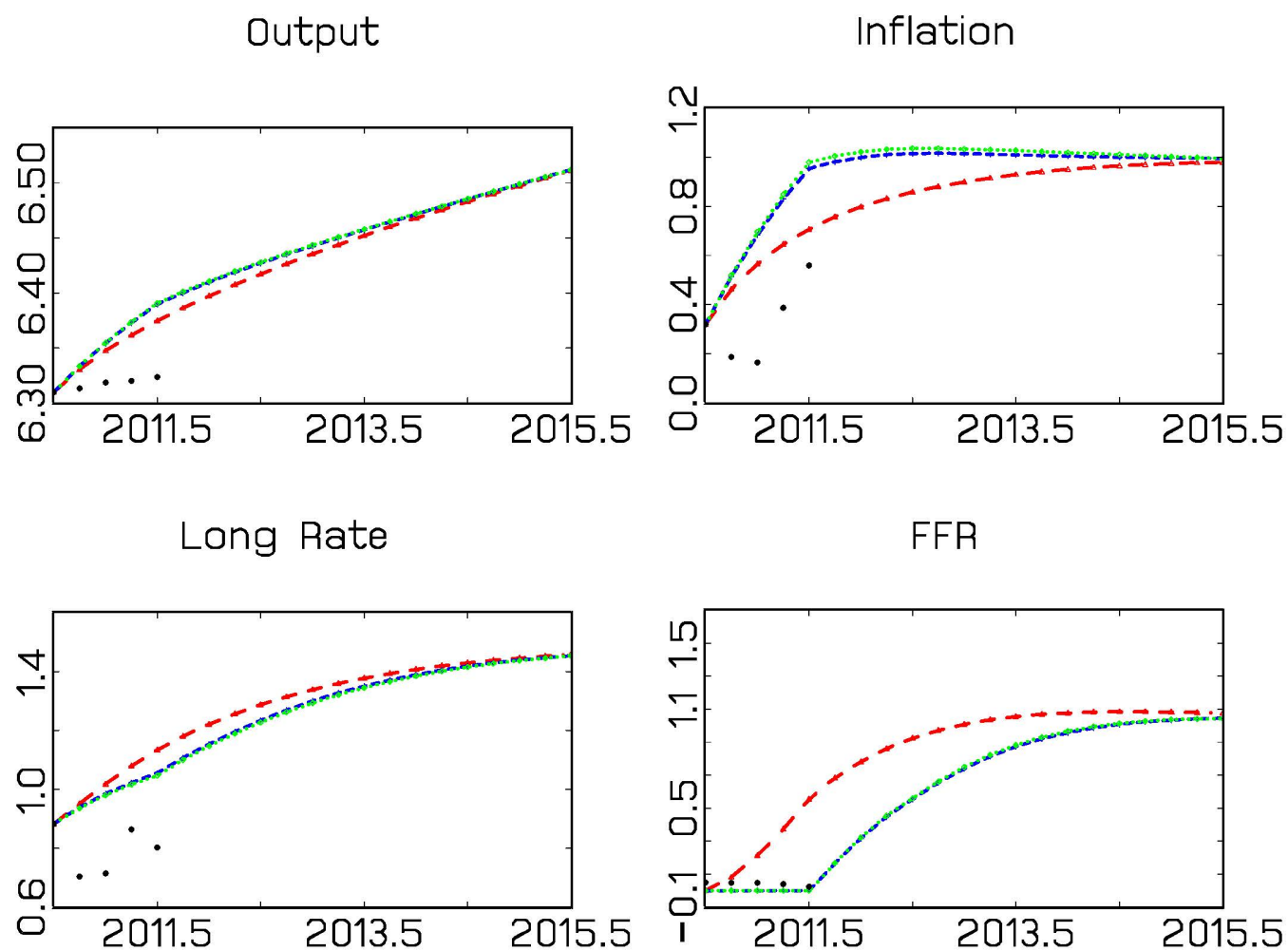


Fig. 9: VAR identified by the exogenous restriction. The red lines show the mean of predicted path of the macro variables without shocks and under no intervention generated by the estimated VAR model using the DSGE exogenous restriction identification. The blue lines show the mean of the predicted path of the macro variables under the LSAPs II generated by the same VAR model. The green lines show the mean of the predicted path of the macro variables under the ZIRP for four quarters generated by the same VAR model.

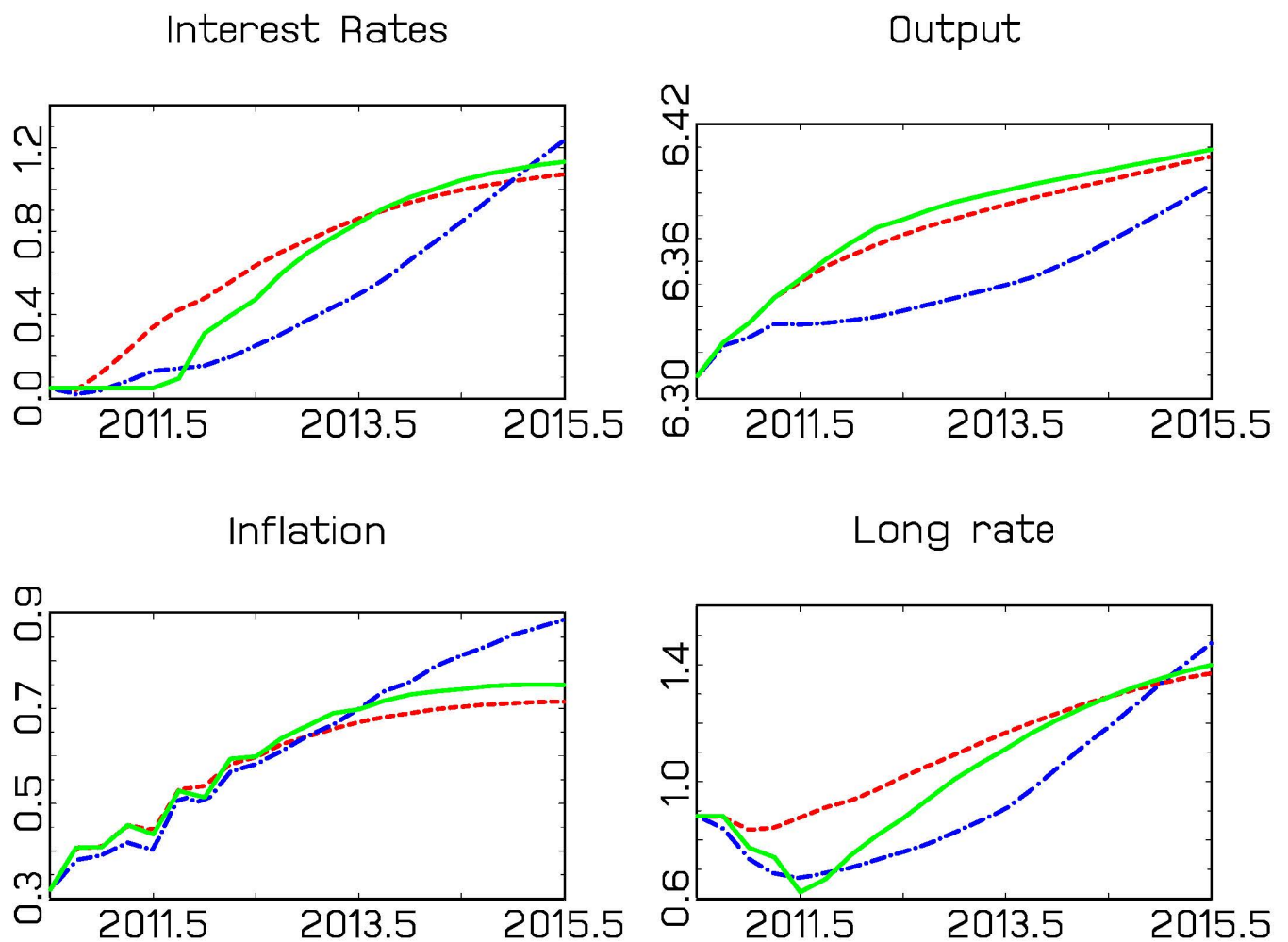


Fig. 10: VAR identified by sign restrictions. The red lines show the mean of predicted paths of the macro variables without shocks and under no intervention generated by the estimated VAR model using the DSGE exogenous restriction identification. The blue lines show the mean of the predicted paths of the macro variables under the LSAPs II generated by the same VAR model. The green lines show the mean of the predicted path of the macro variables under the ZIRP for four quarters generated by the same VAR model. The grey lines are the predictive paths under the combination of the LSAPs and the ZIRP.

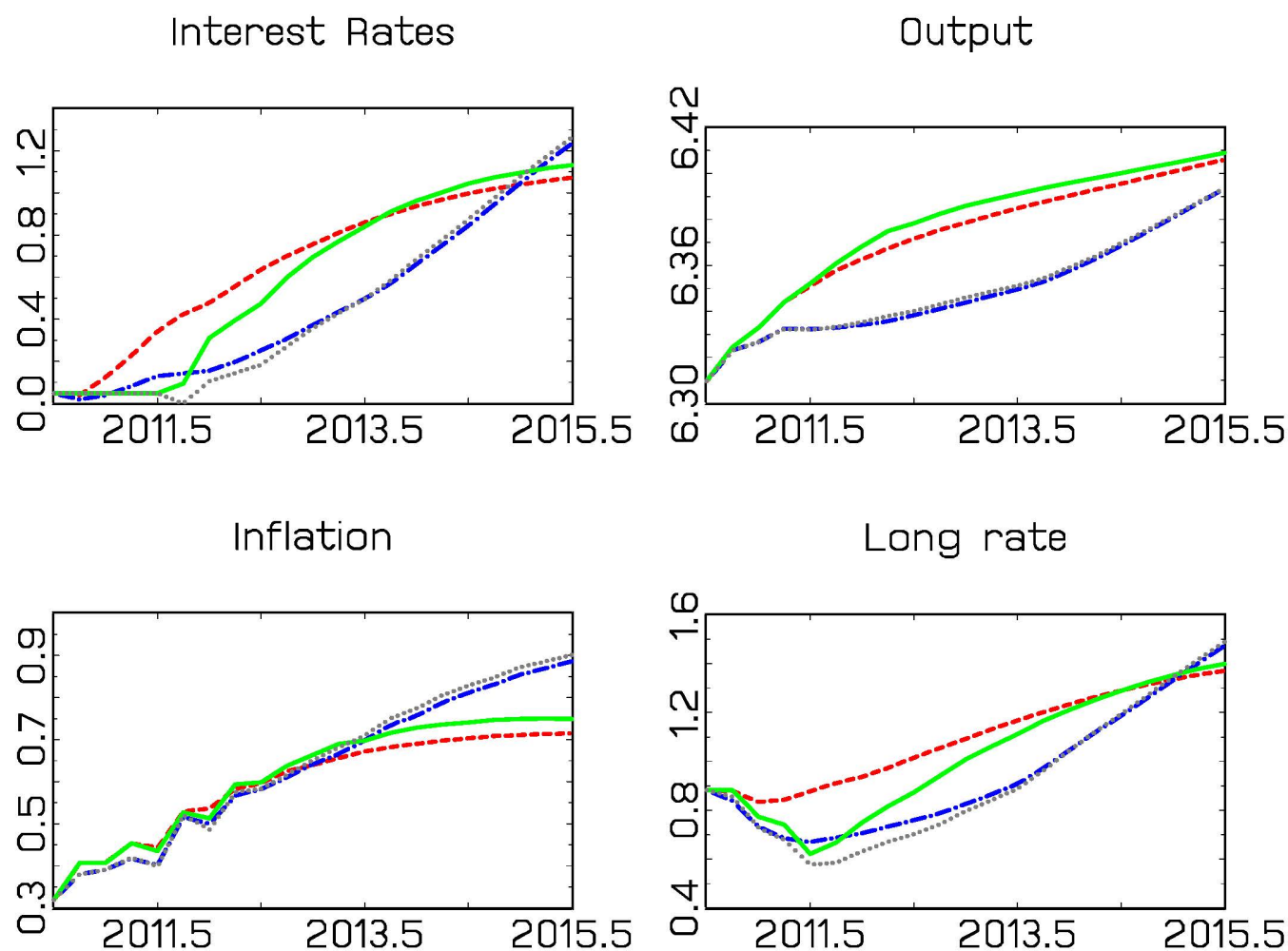


Fig. 11: VAR identified by sign restrictions. The red line shows the mean of predicted path of macro variables without shocks and under no intervention generated by the estimated VAR model using the sign restriction identification. The blue line shows the mean of the predicted path of macro variables under the LSAPs II generated by the same VAR model. The green line shows the mean of the predicted path of macro variables under the ZIRP for four quarters generated by the same VAR model.

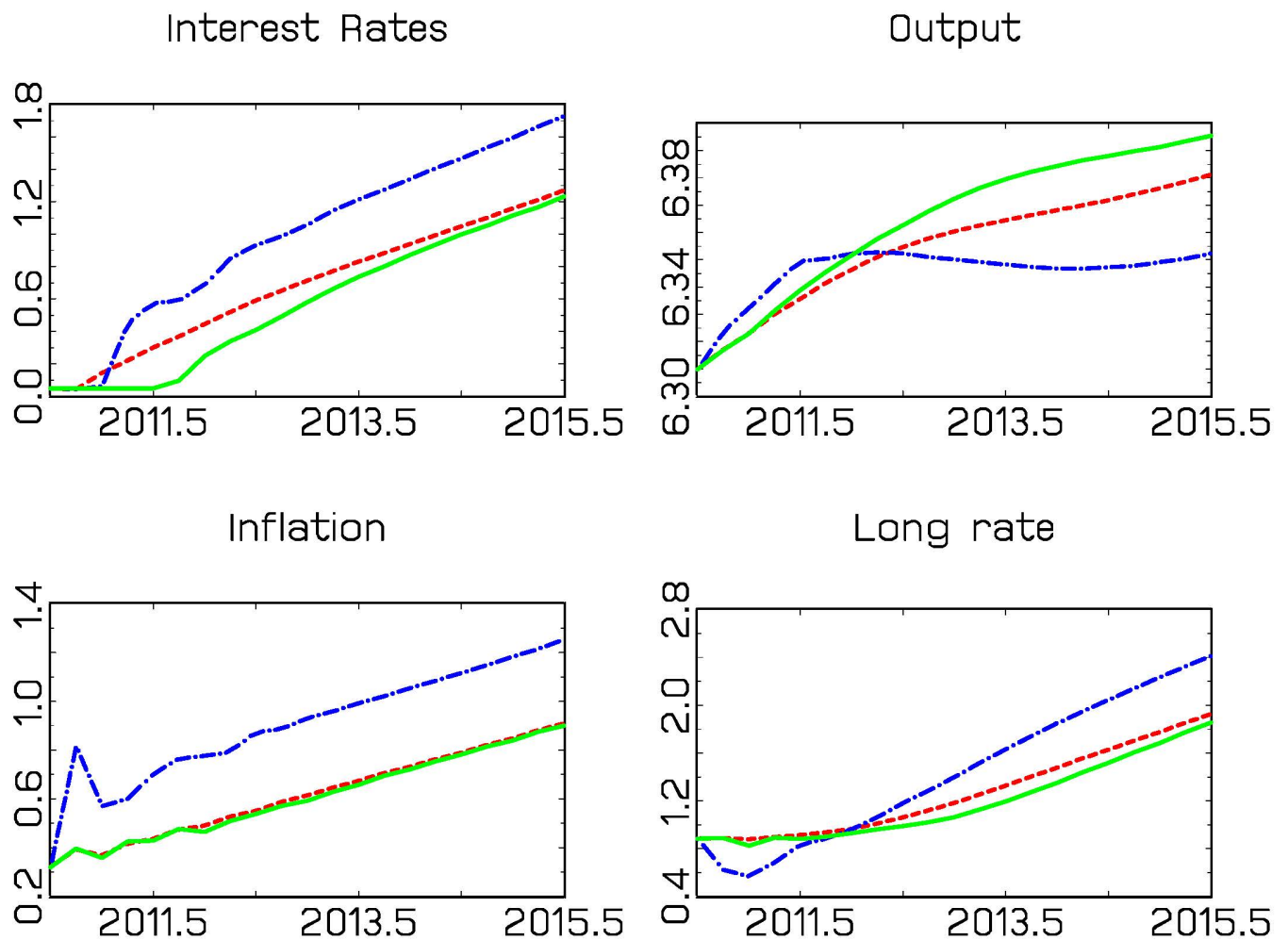


Fig. 12: VAR identified by sign restrictions with identified set. The green line shows the mean of predicted path of macro variables without shocks and under no intervention generated by the estimated VAR model using the sign restriction identification. The red line shows the mean of the predicted path of macro variables under the LSAPs II generated by the same VAR model. The blue line is the identified set of the effects of LSAPs II.

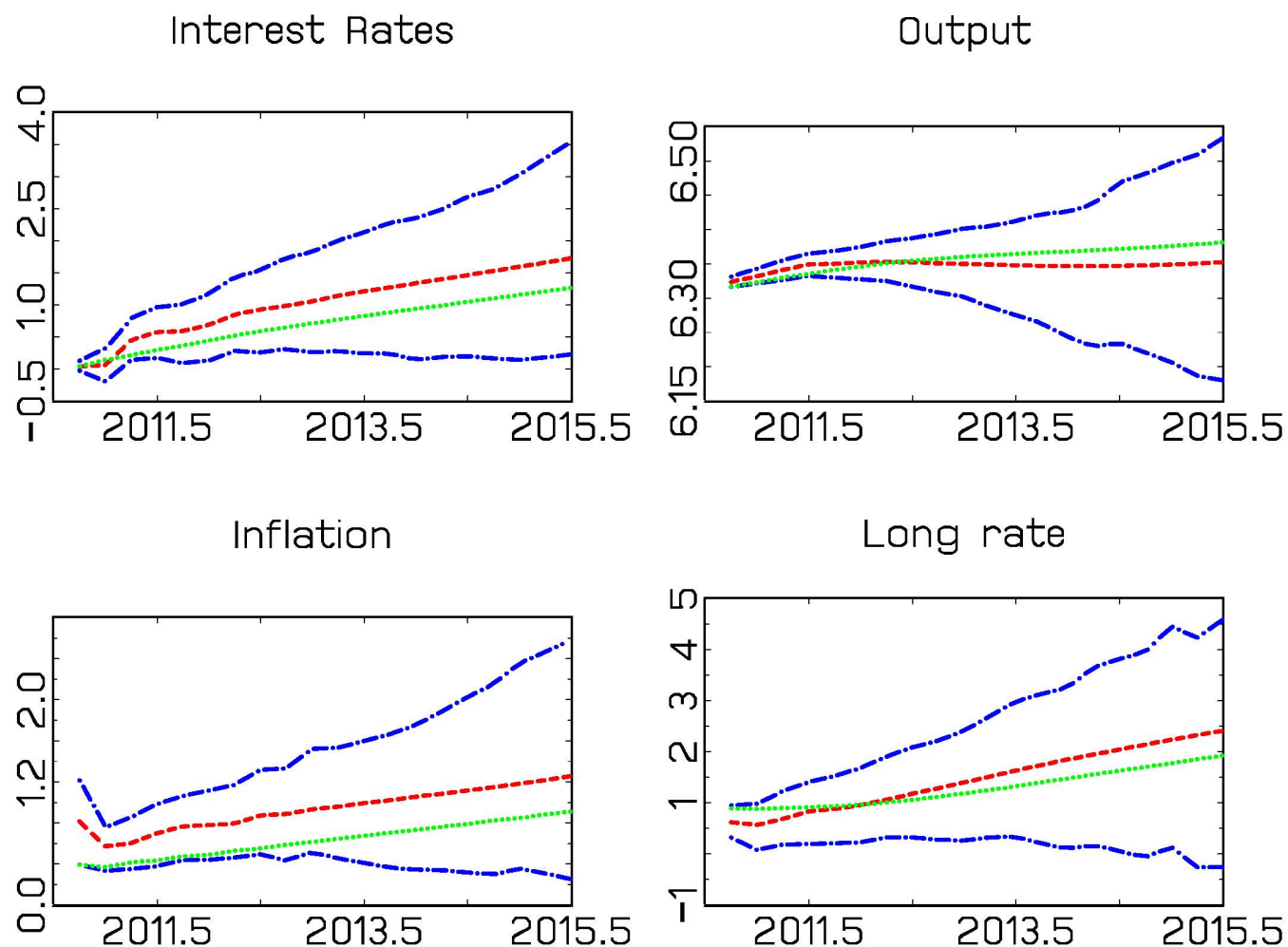


Fig. 13: VAR identified by sign restrictions. The red lines show the mean of predicted paths of macro variables without shocks and under no intervention generated by the estimated VAR model using the sign restriction identification. The blue lines show the mean of the predicted paths of the macro variables under the LSAPs II generated by the same VAR model. The green lines show the mean of the predicted paths of the macro variables under the ZIRP for four quarters generated by the same VAR model. The grey lines are the predictive paths under the combination of the LSAPs and the ZIRP.

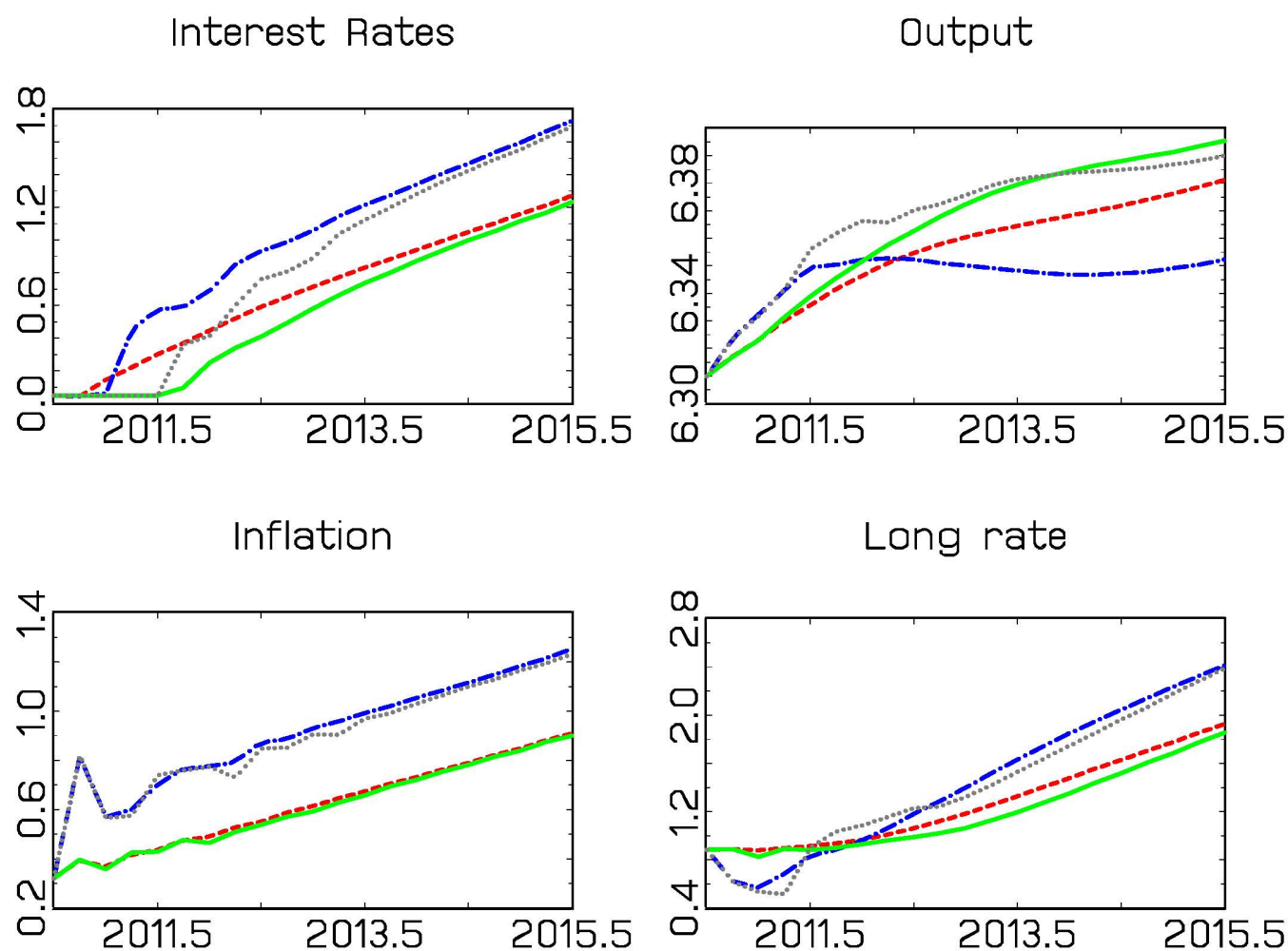


Fig. 14: Summary of effects of LSAPs and ZIRP in DSGE models. The squares stand for mean effects and the circles reflect the uncertainty. Green represents bonds-in-utility model and blue represents the results reported by Chen et al. (2012)

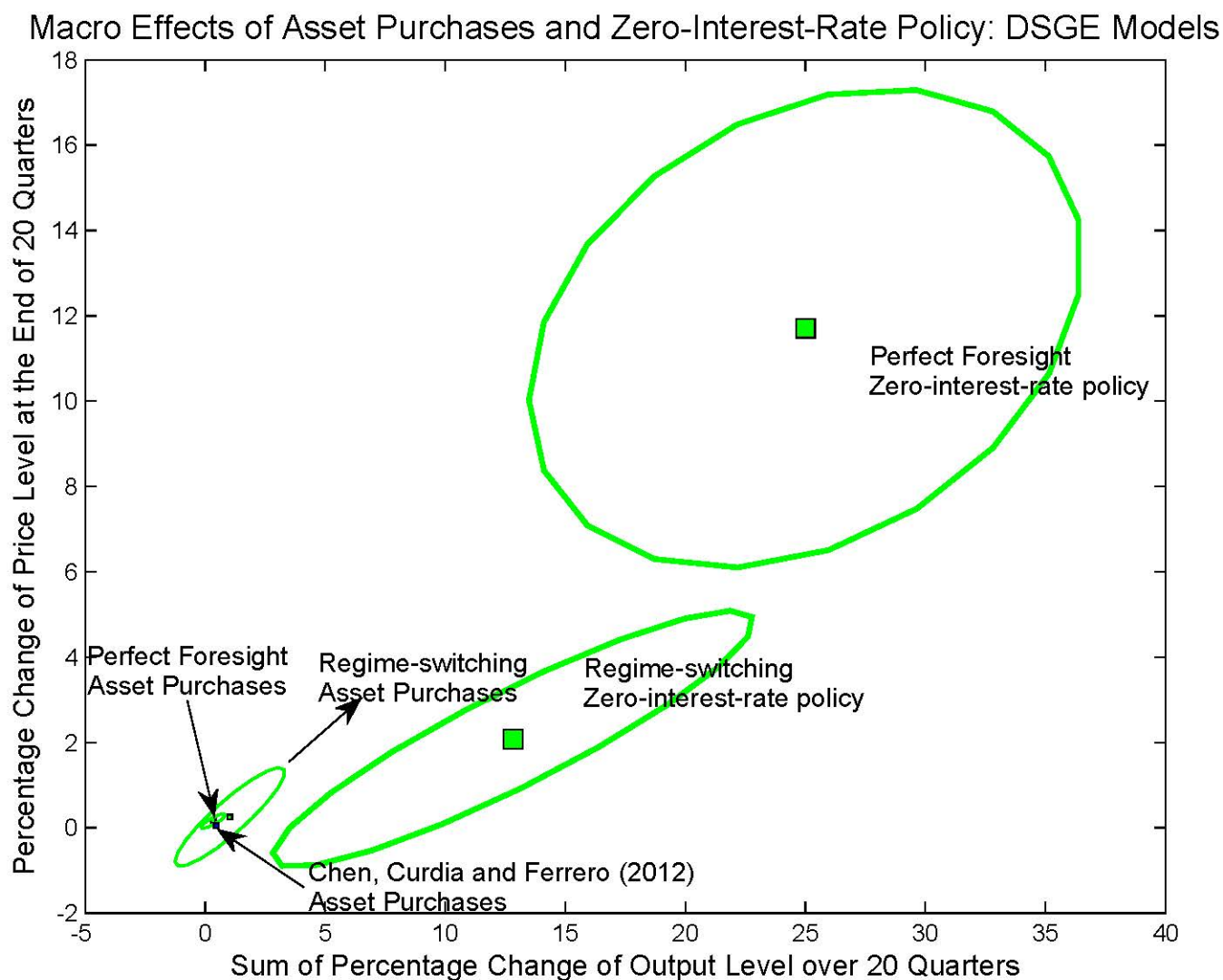
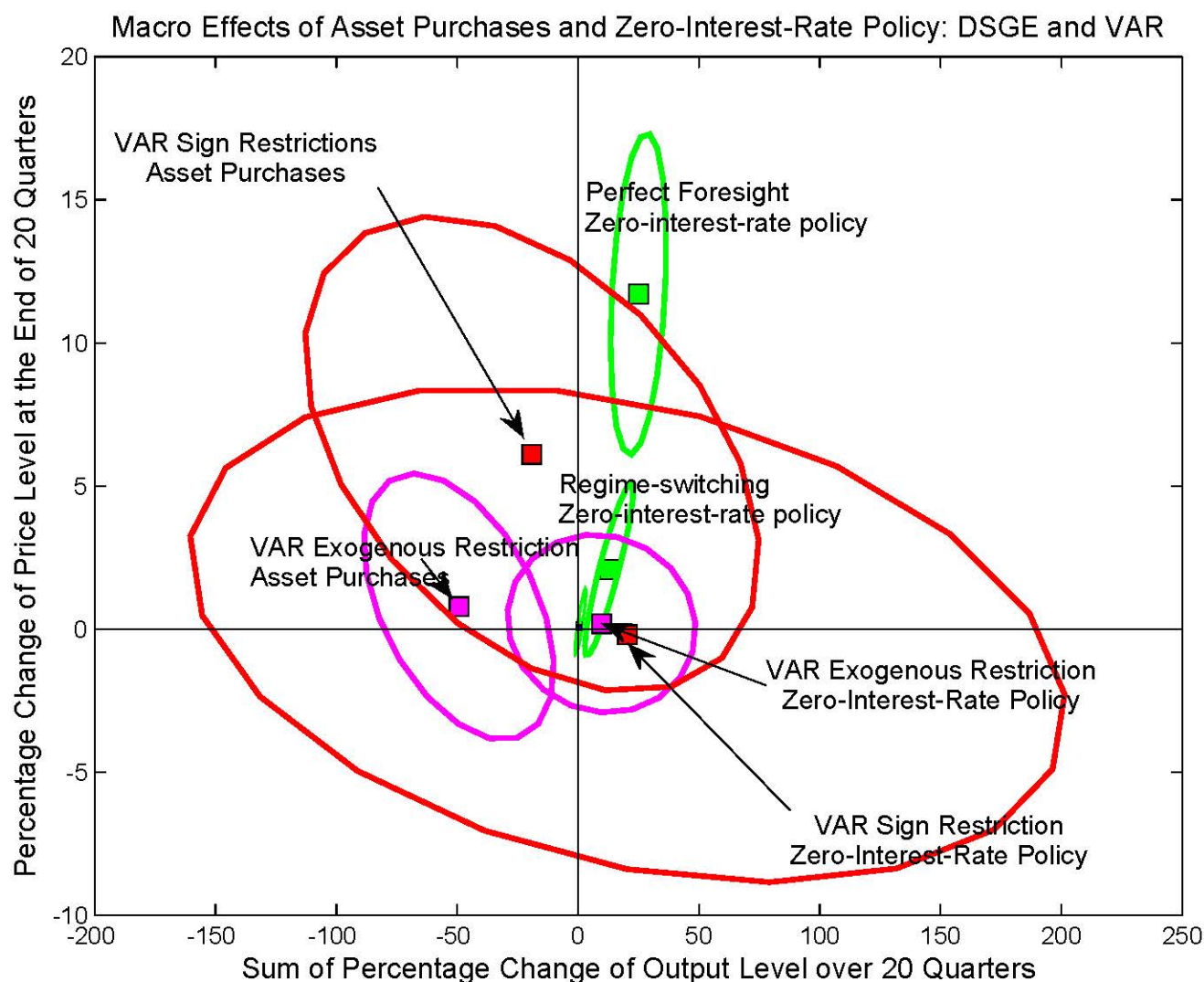


Fig. 15: Summary of effects of LSAPs and ZIRP in DSGE models and VAR models. The squares stand for mean effects and the circles reflect the uncertainty. Green represents bonds-in-utility model, blue represents the results reported by Chen et al. (2012), pink represents the VAR with exogenous restrictions, and red represents the VAR with sign restrictions



8 Appendix: Proof that Schorfheide (2005) and Liu et al. (2011) give rise to the same solution

This section assumes that the only regime-switching parameter is the target steady state interest rate

Schorfheide (2005)

$$\begin{aligned}\hat{R}_t &= \rho_R \hat{R}_{t-1} + (1 - \rho_R) \varphi_\pi \hat{\pi}_t + (1 - \rho_R) \varphi_y \hat{y}_t + \varepsilon_{R,t} + (1 - \rho_R) (1 - \varphi_\pi) \hat{R}_t^* \\ &= \rho_R \hat{R}_{t-1} + (1 - \rho_R) \varphi_\pi \hat{\pi}_t + (1 - \rho_R) \varphi_y \hat{y}_t + \varepsilon_{R,t}^*,\end{aligned}$$

where

$$\varepsilon_{R,t}^* = \varepsilon_{R,t} + (1 - \rho_R) (1 - \varphi_\pi) \left[\log \left(\frac{R_1}{R} \right), \log \left(\frac{R_2}{R} \right) \right] \hat{\mathbf{e}}_{s,t}.$$

Solution by gensys can be written as below where I assume the first shock is $\varepsilon_{R,t}^*$:

$$\begin{aligned}y_t &= \Theta_1 y_{t-1} + \Theta_0 z_t + \Theta_y \sum_{s=1}^{\infty} \Theta_f^s \Theta_z E_t z_{t+s} \\ &= \Theta_1 y_{t-1} + \Theta_0 z_t + (1 - \rho_R) (1 - \varphi_\pi) \Theta_y \sum_{s=1}^{\infty} \Theta_f^s \Theta_z \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \left[\log \left(\frac{R_1}{R} \right), \log \left(\frac{R_2}{R} \right) \right] P^s.\end{aligned}$$

So the constant is

$$\begin{aligned}\Theta_c(K_t) &= (1 - \rho_R) (1 - \varphi_\pi) \Theta_{0.1} \cdot \left[\log \left(\frac{R_1}{R} \right), \log \left(\frac{R_2}{R} \right) \right] \mathbf{e}_{s,t} \\ &= (1 - \rho_R) (1 - \varphi_\pi) \Theta_y \sum_{s=1}^{\infty} \Theta_f^s \Theta_z \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \left[\log \left(\frac{R_1}{R} \right), \log \left(\frac{R_2}{R} \right) \right] P^s \mathbf{e}_{s,t}.\end{aligned}$$

Now I will prove that Liu, Waggnor and Zha (2011) give rise to the same solution.

Assuming the first row of the equilibrium conditions is for the Federal Funds Rate:

$$\begin{bmatrix} \Gamma_0, \begin{pmatrix} -(1-\rho_R)(1-\varphi_\pi) \left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right) \right] \\ 0 \\ \vdots \\ I_2 \end{pmatrix} \end{bmatrix} \begin{bmatrix} y_t \\ \hat{\mathbf{e}}_{s,t} \end{bmatrix} = \begin{bmatrix} \Gamma_1 & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \hat{\mathbf{e}}_{s,t-1} \end{bmatrix} + \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & I_2 \end{bmatrix} \begin{bmatrix} z_t \\ \nu_t \end{bmatrix} + \begin{bmatrix} \Pi\eta_t \\ 0 \end{bmatrix}.$$

Perform QZ decomposition on Γ_0 and Γ_1 and then premultiply both sides by $\begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & I_2 \end{bmatrix}$:

$$\begin{aligned} & \begin{bmatrix} Q_{n \times n} & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} Q'\Lambda Z', \begin{pmatrix} -(1-\rho_R)(1-\varphi_\pi) \left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right) \right] \\ 0 \\ \vdots \\ [0]_{2 \times n} \end{pmatrix} \\ \dots, I_2 \end{bmatrix}_{n \times 2} \begin{bmatrix} y_t \\ \hat{\mathbf{e}}_{s,t} \end{bmatrix} \\ &= \begin{bmatrix} Q_{n \times n} & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} Q'\Omega Z' & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \hat{\mathbf{e}}_{s,t-1} \end{bmatrix} + \begin{bmatrix} Q_{n \times n} & 0 \\ 0 & I_2 \end{bmatrix} \left(\begin{bmatrix} \Psi & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} z_t \\ \nu_t \end{bmatrix} + \begin{bmatrix} \Pi\eta_t \\ 0 \end{bmatrix} \right), \end{aligned}$$

and thus:

$$\begin{aligned} & \begin{bmatrix} \Lambda Z' & Q \begin{pmatrix} -(1-\rho_R)(1-\varphi_\pi) \left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right) \right] \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} y_t \\ \hat{\mathbf{e}}_{s,t} \end{bmatrix} \quad (8.1) \\ &= \begin{bmatrix} \Omega Z' & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \hat{\mathbf{e}}_{s,t-1} \end{bmatrix} + \begin{bmatrix} Q\Psi & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} z_t \\ \nu_t \end{bmatrix} + \begin{bmatrix} Q\Pi\eta_t \\ 0 \end{bmatrix}. \end{aligned}$$

Let $w_t = Z'y_t$, and $w_{t-1} = Z'y_{t-1}$. 8.1 becomes:

$$\begin{aligned} & \Lambda w_t + Q \begin{pmatrix} -(1-\rho_R)(1-\varphi_\pi) \left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right) \right] \\ 0 \\ \vdots \end{pmatrix} \hat{\mathbf{e}}_{s,t} \\ &= \Omega w_{t-1} + Q\Psi z_t + Q\Pi\eta_t, \end{aligned}$$

and thus:

$$\begin{aligned}
& \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} - Q \underbrace{\left(\begin{pmatrix} (1 - \rho_R)(1 - \varphi_\pi) \left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right) \right] \\ 0 \\ \vdots \end{pmatrix} \hat{\mathbf{e}}_{s,t} + \Psi z_t + \Pi \eta_t \right)}_{x_t} \\
&= \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} w_1(t-1) \\ w_2(t-1) \end{bmatrix}.
\end{aligned}$$

Let $M = \Omega_{22}^{-1} \Lambda_{22}$ and solve forward:

$$\begin{aligned}
w_2(t) &= -E_t \left[\sum_{s=1}^{\infty} M^{s-1} \Omega_{22}^{-1} x_2(t+s) \right] \\
&= - \left[\sum_{s=1}^{\infty} M^{s-1} \Omega_{22}^{-1} x_2(t+s) \right].
\end{aligned}$$

Replace x_t with their definition and use the fact $E_t \eta_{t+s} = 0$:

$$\begin{aligned}
&= -E_t \left[\sum_{s=1}^{\infty} M^{s-1} \Omega_{22}^{-1} Q_2 \cdot \left(\Psi Z_{t+s} + \begin{pmatrix} (1 - \rho_R)(1 - \varphi_\pi) \left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right) \right] \\ 0 \\ \vdots \end{pmatrix} \hat{\mathbf{e}}_{s,t+s} \right) \right] \\
&= - \left[\sum_{s=1}^{\infty} M^{s-1} \Omega_{22}^{-1} Q_2 \cdot \left(\Psi Z_{t+s} + \begin{pmatrix} (1 - \rho_R)(1 - \varphi_\pi) \left[\log\left(\frac{R_1}{R}\right), \log\left(\frac{R_2}{R}\right) \right] \\ 0 \\ \vdots \end{pmatrix} \hat{\mathbf{e}}_{s,t+s} + \Pi \eta_{t+s} \right) \right],
\end{aligned}$$

and thus:

$$\begin{aligned}
Q_{2\cdot}\Pi\eta_{t+1} &= \sum_{s=1}^{\infty} \Omega_{22} M^{s-1} \Omega_{22}^{-1} Q_{2\cdot} (\Psi (E_{t+1} z_{t+s} - E_t z_{t+s})) \\
&+ \sum_{s=1}^{\infty} \Omega_{22} M^{s-1} \Omega_{22}^{-1} Q_{2\cdot} \left(\begin{pmatrix} (1 - \rho_R) (1 - \varphi_\pi) [\log (\frac{R_1}{R}), \log (\frac{R_2}{R})] \\ 0 \\ \vdots \end{pmatrix} (E_{t+1} \hat{\mathbf{e}}_{s,t+s} - E_t \hat{\mathbf{e}}_{s,t+s}) \right).
\end{aligned}$$

If the solution is unique:

$$Q_{1\cdot}\Pi = \Phi Q_{2\cdot}\Pi.$$

Premultiplying 8.2 by $[I - \Phi]$:

$$\begin{aligned}
&\begin{bmatrix} \Lambda_{11} & \Lambda_{12} - \Phi \Lambda_{22} \\ 0 & I \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \\
&- \begin{bmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{bmatrix} \left(\begin{pmatrix} (1 - \rho_R) (1 - \varphi_\pi) [\log (\frac{R_1}{R}), \log (\frac{R_2}{R})] \\ 0 \\ \vdots \end{pmatrix} \hat{\mathbf{e}}_{s,t} \right) \\
&= \begin{bmatrix} \Omega_{11} & \Omega_{12} - \Phi \Omega_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1(t-1) \\ w_2(t-1) \end{bmatrix} + \begin{bmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{bmatrix} \Psi z_t \\
&- \begin{bmatrix} 0 \\ E_t [\sum_{s=1}^{\infty} M^{s-1} \Omega_{22}^{-1} x_2(t+s)] \end{bmatrix}.
\end{aligned}$$

Finally,

$$\begin{aligned}
& y_t + Z \begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1} (\Lambda_{12} - \Phi \Lambda_{22}) \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{bmatrix} \left(\begin{pmatrix} -(1 - \rho_R) (1 - \varphi_\pi) [\log(\frac{R_1}{R}), \log(\frac{R_2}{R})] \\ 0 \\ \vdots \end{pmatrix} \hat{\mathbf{e}}_{s,t} \right) \\
&= Z \begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1} (\Lambda_{12} - \Phi \Lambda_{22}) \\ 0 & I \end{bmatrix} \begin{bmatrix} \Omega_{11} & \Omega_{12} - \Phi \Omega_{22} \\ 0 & 0 \end{bmatrix} Z' y_{t-1} \\
&\quad + Z \begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1} (\Lambda_{12} - \Phi \Lambda_{22}) \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{bmatrix} \Psi z_t \\
&\quad - Z \begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1} (\Lambda_{12} - \Phi \Lambda_{22}) \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ E_t [\sum_{s=1}^{\infty} M^{s-1} \Omega_{22}^{-1} Q_{2\cdot} \Psi z_{t+s}] \end{bmatrix} \\
&\quad - Z \begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1} (\Lambda_{12} - \Phi \Lambda_{22}) \\ 0 & I \end{bmatrix} \\
&\quad \left[E_t \left[\sum_{s=1}^{\infty} M^{s-1} \Omega_{22}^{-1} Q_{2\cdot} \begin{pmatrix} 0 \\ (1 - \rho_R) (1 - \varphi_\pi) [\log(\frac{R_1}{R}), \log(\frac{R_2}{R})] \\ 0 \\ \vdots \end{pmatrix} \hat{\mathbf{e}}_{s,t+s} \right] \right].
\end{aligned}$$

By simplifying notation, I can rewrite the above equation as:

$$\begin{aligned}
y_t &= \Theta_1 y_{t-1} + \Theta_0 \left(\Psi z_t + \begin{pmatrix} (1 - \rho_R) (1 - \varphi_\pi) [\log(\frac{R_1}{R}), \log(\frac{R_2}{R})] \\ 0 \\ \vdots \end{pmatrix} \hat{\mathbf{e}}_{s,t} \right) \\
&\quad + \Theta_y \sum_{s=1}^{\infty} \Theta_f^{s-1} \Theta_z E_t \left(\Psi z_{t+s} + \begin{pmatrix} (1 - \rho_R) (1 - \varphi_\pi) [\log(\frac{R_1}{R}), \log(\frac{R_2}{R})] \\ 0 \\ \vdots \end{pmatrix} \hat{\mathbf{e}}_{s,t+s} \right),
\end{aligned}$$

where

$$\Theta_1 = Z \begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1} (\Lambda_{12} - \Phi \Lambda_{22}) \\ 0 & I \end{bmatrix} \begin{bmatrix} \Omega_{11} & \Omega_{12} - \Phi \Omega_{22} \\ 0 & 0 \end{bmatrix} Z',$$

$$\begin{aligned}
\Theta_0 &= Z \begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1} (\Lambda_{12} - \Phi \Lambda_{22}) \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_{1\cdot} - \Phi Q_{2\cdot} \\ 0 \end{bmatrix}, \\
\Theta_y &= -Z \begin{bmatrix} \Lambda_{11}^{-1} & \Lambda_{11}^{-1} (\Lambda_{12} - \Phi \Lambda_{22}) \\ 0 & I \end{bmatrix}, \\
\Theta_f &= M,
\end{aligned}$$

and

$$\Theta_z = \Omega_{22}^{-1} Q_{2\cdot}.$$

This is exactly the same as treating $\hat{\mathbf{e}}_{s,t+s}$ as a shock as in Schorfheide (2005).