A Dynamic Model of Leverage and Interest Rates*

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First Version: October 1, 2013
This Version: October 25, 2013

Abstract

I study a dynamic model where both leverage and interest rates are determined in equilibrium. The model considers risk neutral agents, or equivalently agents with an infinite elasticity of inter-temporal substitution. In this setting without financial frictions the risk-free rate is given by the discount rate of agents. However, I show that the introduction of collateral constraints creates a wedge between the equilibrium risk-free rate and agents' rate of discount. The collateral constraint limits arbitrage and some agents enjoy risk-adjusted returns that are higher than the equilibrium risk-free rate. In a dynamic setting the possibility of enjoying these excess returns in the future makes agents willing to lend today at lower risk-free rates relative to their rate of preferences. Excess returns are expected to be larger when leverage is expected to be lower, so a more severe expected collapse in leverage increases the wedge between the rate of preferences and the equilibrium risk-free rate, i.e. reduces the risk-free rate. At the same time, the model predicts that this will increase risk premiums on debts that default when leverage is expected to be low.

* I specially thank my advisor John Geanakoplos for his advise and constant encouragement. I also benefited from helpful discussions with Mikhail Golosov, Andrew Metrick, and Guillermo Ordoñez, and comments from Alexandros Vardoulakis and participants in seminars at Yale's Macro Lunch and Central Bank of Chile. The opinions expressed do not necessarily reflect those of the Federal Reserve Board or its staff.

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1 introduction

After the recent financial crises of 2008 the role of leverage fluctuations for economic dynamics has received renewed attention. Most dynamic models where agents leverage their asset purchases, for tractability make special assumptions making either leverage or the interest rate effectively fixed. Therefore, understanding how leverage fluctuations affect interest rates is a largely unexplored question. This paper analyse a dynamic model where both leverage and the interest rates are endogenously determined and exhibit interesting dynamics. The analysis is used to evaluate the effect of the collapse in leverage during periods of financial turmoil on equilibrium interest rates.

As a benchmark I consider a model with risk neutral agents, and therefore an infinite elasticity of inter-temporal substitution. In this setting, in the absence of financial frictions in equilibrium the risk-free rate needs to be equal to the preference discount rate of agents. However, as I show the introduction of financial frictions creates a wedge between the risk-free rate and the rate of preferences. I consider that borrowing is limited by a collateral constraint, so arbitrage is limited and some agents enjoy risk-free returns higher than the equilibrium interest rate. In a dynamic setting the possibility of agents to enjoy this excess returns in the future makes them willing to lend at an interest rate that is lower than their rate of preferences.

More interestingly, as the future collapse in leverage becomes more severe, the excess returns that agents enjoy during the period of financial turmoil will be even larger, incentivizing agents to lend at even lower interest rates today. As the change in interest rates is due to the relative increase in the marginal value of funds during the turmoil, risky debt that defaults during the turmoil will carry a higher risk premium. So the model explains both the decline in the itnerest rate and the increase in risk premiums when the collapse in leverage is expected to become more severe.

Relation to the Literature. The literature that studies the effect of financial frictions for macroeconomic dynamics shows how these frictions can generate persistence and amplification of economic shocks (Bernanke and Gertler, 1989; Kiyotaki and Moore, 1997; Bernanke, Gertler and Gilchrist, 1999). On the one hand, transitory shocks can lead to persistent effects as they affect the net-worth of leveraged agents and net-worth takes tome to rebuild. On the other hand, adverse shocks can be amplified through the financial system when there are financial frictions. An adverser shock leads to a drop in prices and net-worth of leveraged agents, the decline in net-worth feedsback into a further decline of asset prices further reducing net-worth and giving rise to an adverse feedback loop.

For tractability most models make special assumptions so either leverage or the interest rate have a non-interesting dynamics. In one set of models the risk-free rate is determined by the rate of preference of individual agents, abstracting from the dynamics of this variable (Kiyotaki and Moore, 1997; Geanakoplos, 1997, 2003; Brunnermeier and Sannikov, 2012). In other model the borrowing constraint is such that leverage is determined by one of the parameters of the model (e.g. Jeanne and Korineck, 2010).

Some models like Bernanke, Gertler and Gilchrist (1999) (henceforth BGG) have an interesting dynamics of both leverage and interest rates. But in BGG the leverage and risk premium are proportional, so the risk premium is lowest when leverage collapses. In addition the structure of the BGG model imply that the equilibrium risk-free rate is determined by the consumption-saving decision of households complicating the analysis between the relantionship of leverage and the risk-free rate. A related litrature looks at the relantionship between equilibirum interest rates and leverage in house purchases (see Jeske, Krueger and Mitman, 2012 and references therein).

The rest of the paper is organized as follows. I present the model in section 2. In section 3, I study the entrepreneurs' problem and describe their optimal behavior, given their portfolio choices. In section 4 I characterize entrepreneurs' portfolio choices, asset prices and the aggregation result that renders irrelevant the distribution of net-worth and reduce the dimensionality of the problem to solve for equilibrium. I show how to solve for equilibrium using a numerical calibration and describe the equilibrium in the baseline case in section 5. In section 6 I present comparative static exercises and present the main result of the paper. Finally, I give some concluding remarks in section 7.

2 The Model

There are 3 periods: $t = 0, 1, 2$. There are two goods: capital and a perishable consumption good, or simply fruit. Entrepreneurs use capital to produce fruit, and entrepreneurs differ in their productivities of using capital.

Capital has an exogenous stochastic resale value in period $t = 2$. These values are the only source of aggregate uncertainty in the model and can be represented in a binomial tree as depicted in Figure 1. The value of capital in the terminal state can take four different values p_{UU}, p_{UD}, p_{DU} and p_{DD}. In period $t = 1$ the aggregate state can be either U or D, representing information about the value of capital in the final period. After state U only final prices p_{UU} or p_{UD} may obtain, whereas after state D only prices p_{DU} or p_{DD} may obtain. In period $t = 0$ the state is 0. The probability of moving "up" in the tree of events, i.e., of state U following state 0, of state UU following U, and of state DU following state D, is given by π.

To denote the possible states in the tree of events let me introduce the following notation. Let

$s \in \{0, U, D, UU, UD, DU, DD\}$ denotes the aggregate state, and let $t(s)$ be a mapping from the state to the time period. Thus, for instance, $t(D) = 1$ and $t(DU) = 2$. The possible successors of state s will be denoted the random variable s'. In addition, the history followed by state U will be denoted sU (with sD analogously defined). Moreover, the predecessor of state s will be denoted s^*. Finally, let x_s denote the value of variable x in state s, and let $x_{t(s)}$ denotes the random variable of possible values of variable x in period $t(s)$.

Entrepreneurs have identical preferences for consumption and are risk-neutral. Future consumption is discounted at rate β. So preferences are represented by the utility function

$$U(c_0, c_1, c_2) = \sum_{t=0}^{2} \beta^t \mathbb{E}_0[c_t] = c_0 + \beta \mathbb{E}_0[c_1] + \beta^2 \mathbb{E}_0[c_2] \tag{1}$$

Entrepreneurs can use capital to produce fruit using the following technology. At the beginning of the period, in state s, entrepreneurs receive a random idiosyncratic productivity shock $\theta_s \in \Theta = [\underline{\theta}, \bar{\theta}]$. After learning their idiosyncratic productivity, entrepreneurs choose their land holdings k_s. In the next period, this land produces $y_s = \theta_s k_s$ units of fruit, in any future state of the world.[1] Capital depreciates at a rate δ.

For tractability and to facilitate the intuition it is assumed that produced output is the same in every state of the world. That is, production is a riskless activity, thus the portfolio problem of entrepreneuers will be more tractable, and returns to production can be directly compared to the riskless rate facilitiating the intuition.

Entrepreneurs, then, can be identified by their history of productivity shocks, $\theta = (\theta_0, \theta_1, \ldots, \theta_t)$, where θ_t is the productivity in period t. In line with the notation for the aggregate state let me use θ' and θ^* to denote the history of productivity shocks up to the next and the previous period, respectively. Entrepreneurs productivities are drawn from a distribution with continuous CDF $F(\theta)$ and $dF(\theta)$ denote the measure of entrepreneurs with productivity θ. I assume that the idiosyncratic productivity θ_t is independent of previous productivity shocks, i.e.,

$$dF(\theta) = dF(\theta_0) dF(\theta_1) \cdots dF(\theta_t)$$

In this setting with heterogeneous agents, the *state space* is described by the aggregate state s and the joint distributions of idiosyncratic productivities and the holdings from last period of

[1] Note that it is assumed that production is the same in every future state of the world, but the model can be extended to consider aggregate production risk. For instance, let $a_{s'}$ be the stochastic level of aggregate productivity depending on the state of the world in the next period s', and consider that land produces $a_{s'} \theta k_s$.

capital, debt contracts and fruit $(\{J_{k,s}(k_{s^*}, \theta), J_{\phi,s}(b_{\phi,s^*}, \theta), J_{y,s}(y_s, \theta)\})$.[2,3] To simplify the notation burden let me use $k_s(\theta)$ to denote $k(\theta; s, J_{k,s}(k_{s^*}, \theta), J_{\phi,s}(b_{\phi,s^*}, \theta), J_{y,s}(y_s, \theta))$.

In addition to the fruit-production technology entrepreneurs have access to an investment technology that turn fruit into capital. In state s an entrepreneur can produce i_s units of capital at a cost of $\Psi\left(\frac{i_s}{n_s}\right)n_s$, in terms of fruit. Ψ is an increasing and convex investment-cost function.[4] For the numerical exercises considered below, it is assumed that $\Psi\left(\frac{i_s}{n_s}\right) = \alpha_1\left(\frac{i_s}{n_s}\right) + \alpha_2\left(\frac{i_s}{n_s}\right)^2$, with $\alpha_1 \geq 0$ and $\alpha_2 > 0$. So entrepreneurs in each state solve the following static profit maximization problem

$$\max_{i_s} p_s i_s - \Psi\left(\frac{i_s}{n_s}\right)n_s \tag{2}$$

$$\text{s.t.} \qquad i_s \geq 0 \qquad (\eta_{i,s})$$

Debt Contracts and Collateral. Entrepreneurs can borrow using a menu of different debt contracts. Available debt contracts are elements of the finite set Φ. These debt contracts specify promised deliveries in successor states $\phi \in \mathbb{R}_+^2$ and require to pledge one unit of capital as collateral. Note that there is no loss of generality in normalizing contracts to have collateral equal to one unit of capital, as there is one degree of freedom in describing these contracts: a contract making twice as big a promise and backed by twice as much collateral will trade for twice the price. To simplify the exposition I restrict attention to uncontingent promises $\phi \in \Phi \subset \mathbb{R}_+$, so debt contracts are identified by ϕ. As I discuss below in the unique equilibrium the effective payoffs of traded contracts will be state contingent. So there is no loss of generality in restricting attention to uncontingent contracts.

To handle the possibility that effective deliveries differ from the promised amounts I introduce the following notation. Let $d(\phi, s')$ be the effective delivery, if the future aggregate state is s', of a contract that promise ϕ. Then the effective deliveries are given by,

$$d(\phi, s') = \min\{\phi, (1-\delta)p_{s'}\}$$

that is the minimum between the promised delivery and the value of a unit of land in state s'. In

[2] In period 0 the joint distribution of idiosyncratic productivities and endowments of capital and fruit.

[3] As I show below, the individual decision problem depends on the value of net-worth at the beginning of the period, so the state space can be simplified to include only the joint distribution of idiosyncratic productivities and net-worth $J_s(n_s, \theta)$.

[4] Note that the capital adjustment cost is proportional to net-worth at the beginning of state s, n_s. This assumption is made for tractability. An alternative specification would be that the capital adjustment cost is proportional to the capital stock before capital production, say $\tilde{k}_s$, or proportional to the capital remaining from the previous period: $(1-\delta)k_{s^*}$.

this environment, issuing a ϕ-promise is qualitatively different than buying a ϕ-promise. To handle this difference without additional notation I use the sign of the holding of ϕ-promises to distinguish these cases. Let b_ϕ be the holdings of ϕ-promises, then b_ϕ when $b_\phi > 0$ denotes the total purchases of this promise, whereas $|b_\phi|$ when $b_\phi < 0$ denotes the total issuance of this promise. Note that $|b_\phi|$ when $b_\phi < 0$, corresponds to the negative part function, denoted by $(b_\phi)_-$. As each promise issued needs to backed by one unit of capital, for any given agent in any state s, it must be that

$$\sum_{\phi \in \Phi} (b_{\phi,s})_- \leq k_s \tag{3}$$

This is the *collateral constraint*.

To describe the constraints on entrepreneurs' choices it is useful to consider the value of their net-worth in state s, that I denote by n_s. Entrepreneurs start in period 0 with net-worth equal to the value of their endowment

$$n_0(\theta_{-1}) = e_c(\theta_{-1}) + p_0 e_k(\theta_{-1})$$

where $e_c(\theta_{-1})$ and $e_k(\theta_{-1})$ are the distribution of initial endowments of fruit and capital, which are assumed independent of the distribution of productivity shocks in state 0. The *evolution of net-worth* is governed by entrepreneurs' portfolio choices and it is given by

$$n_{s'}(\theta) \leq \theta_s k_s(\theta) + p_{s'}(1-\delta)k_s(\theta) + \sum_{\phi \in \Phi} b_{\phi,s}(\theta) d(\phi, s') \tag{4}$$

On the other hand, in every state entrepreneurs consumption, portfolio and investment choices are constrained by the value of their net-worth and their investment decision,

$$c_s(\theta) + p_s k_s(\theta) + \sum_{\phi \in \Phi} q_{\phi,s} b_{\phi,s}(\theta) + \Psi\left(\frac{i_s(\theta)}{n_s}\right) n_s \leq n_s + p_s i_s(\theta) \tag{5}$$

This is the *current period budget constraint*. Finally, entrepreneurs are constrained to hold non-negative capital holdings and consumption in every state.

$$k_s(\theta), c_s(\theta) \geq 0 \tag{6}$$

Definition 1 (Collateral Competitive Equilibrium *CCE*) *A CCE consists of prices* $\{p_s,\ \{q_{\phi,s}\}_{\phi\in\Phi}\}_{s=0,U,D}$ *and* $dF(\theta)$*-measurable functions for allocations* $\{k_s(\theta),\ \{b_{\phi,s}(\theta)\}_{\phi\in\Phi},\ i_s(\theta)\}_{s=0,U,D}$ *and* $\{c_s(\theta)\}_{s=0,U,D,UU,UD,DU,DD}$ *such that entrepreneurs' behavior is optimal, and in all states markets clear for consumption, capital and every debt contracts.*

Remark 1 *By Walras' Law for incomplete markets, market clearing for consumption in each state are redundant.*

The existence of *CCE* was established by Geanakoplos and Zame (2002, 2007). In the next section I study the entrepreneurs' decision problem, and in the following section I turn to the characterization of this equilibrium.

3 Entrepreneurs' Decision Problem

The entrepreneurs' decision problem can be divided into a static capital investment problem and a dynamic consumption-portfolio problem. Consider an entrepreneur that arrives at state s with net-worth s. She will choose the optimal capital investment to maximize the value of the new capital stock net of its production cost,

$$\max_{i_s} p_s i_s - \Psi\left(\frac{i_s}{n_s}\right) n_s \qquad \text{s.t.} \quad i_s \geq 0$$

Then, the optimal investment policy is given by

$$i_s^* = \Psi'^{-1}(p_s) n_s$$

Note that since Ψ is convex, Ψ' and, thus, Ψ'^{-1} are increasing. So investment is higher when the price of land is higher. It also follows from the optimal investment policy that investment is higher when net-worth is higher.

The optimal investment policy can be used to simplify the current period budget constraint (5)

$$c_s(\theta) + p_s k_s(\theta) + \sum_{\phi \in \Phi} q_{\phi,s} b_{\phi,s}(\theta) \leq \left\{1 + p_s \Psi'^{-1}(p_s) - \Psi\left(\Psi'^{-1}(p_s)\right)\right\} n_s \tag{7}$$

Let $V(\theta, n_s; s)$ be the indirect utility of net-worth in state s for an entrepreneur with productivity $\theta = (\theta_0, \ldots, \theta_s)$, given by

$$V(\theta, n_s; s) = \max \mathbb{E}_s \left[\sum_{j=t(s)}^{2} \beta^{j-t(s)} c_j(\theta_j)) \right] \tag{8}$$

subject to the budget constraints (3), (4), (6) and (7).[5]

[5] Note that the state determines both the time period, changing time preferences, and the information available to entrepreneur.

Lemma 1 (Value Function Characterization) *If entrepreneurs are risk neutral, the value function is linear in net-worth, i.e.,*

$$V(\theta, n_s; s) = a_s(\theta_s)n_s$$

This imply that the history of idiosyncratic shocks is irrelevant up to the current value of net-worth, and allows to define the value function as a function of only contemporaneous productivity and contemporaneous net-worth, $V(\theta_s, n_s; s) = a_s(\theta_s)n_s$. In addition, the coefficients of the value function $a_s(\theta_s)$ are: (i) strictly positive, i.e., $a_s(\theta_s) > 0$ for any state s and productivity θ_s; (ii) non-decreasing function of entrepreneur's productivity; and (iii) characterized recursively by

$$a_s(\theta_s) = \left\{1 + p_s\Psi'^{-1}(p_s) - \Psi\left(\Psi'^{-1}(p_s)\right)\right\} \max\{1, \beta\mathbb{E}_s[a_{s'}(\theta_{s'})R^*(\theta_s; s')]\} \tag{9}$$

where $R^*(\theta_s; s')$ is the random return of the optimal portfolio chosen by entrepreneur θ_s in state s.

Moreover, the optimal policies are linear in net-worth, and the optimal policy for consumption satisfies

$$c_s(\theta_s) = 0 \qquad \textit{if} \quad \beta\mathbb{E}_s\left[a_{s'}(\theta_{s'})R^*(\theta_s; s')\right] > 1 \tag{10}$$

Proof of Lemma 1: *Consider, in state s, two entrepreneurs A and B with the same idiosyncratic productivity θ_s, and net-worth n_s^A and n_s^B, respectively. Let*

$$V^A(\theta_s, n_s^A; s) = \max \mathbb{E}_s\left[\sum_{j=t}^{2}\beta^{j-t}c_j^A(\theta_j)\right] \qquad \textit{and} \qquad V^B(\theta_s, n_s^B; s) = \max \mathbb{E}_s\left[\sum_{j=t}^{2}\beta^{j-t}c_j^B(\theta_j)\right]$$

Then, we want to show that

$$\frac{V^A(\theta_s, n_s^A; s)}{n_s^A} = \frac{V^B(\theta_s, n_s^B; s)}{n_s^B}$$

By contradiction, suppose that $V^A(\theta_s, n_s^A; s)/n_s^A > V^B(\theta_s, n_s^B; s)/n_s^B$. Let $k_z^A(\theta_z)$, $c_z^A(\theta_z)$, $b_{\phi,z}^A(\theta_z)$, $i_z^A(\theta_z)$ be agent's A optimal policies, for any state z that follows state s and productivity θ_z that may obtain, i.e.

$$V^A(\theta_s, n_s^A; s) = \mathbb{E}_s\left[\sum_z \beta^{t(z)}c_z^A(\theta_z)\right]$$

Lets construct a feasible policy for agent B that achieves an indirect utility of at least $\frac{n_s^B}{n_s^A}V^A(\theta_s, n_s^A; s)$. Let $\zeta = \frac{n_s^B}{n_s^A}$ and consider the policy $\zeta k_z^A(\theta_z)$, $\zeta c_z^A(\theta_z)$, $\zeta b_{\phi,z}^A(\theta_z)$, $\zeta i_z^A(\theta_z)$ for any state z that follows state s and productivity θ_z that may obtain.

Recall that from the solution to the static investment problem we had that $i_s^l = \Psi'^{-1}(p_s)n_s^l$ for $l = A, B$. So i_z^B is optimal for B. In addition, the ζ-policy is feasible for agent B, as by definition $n_s^B = \zeta n_s^A$, and the current period budget constraint, the evolution of net-worth, and the collateral constraint, are homogeneous of degree 1 in $k_z(\theta_z)$, $c_z(\theta_z)$, $b_{\phi,z}(\theta_z)$, $n_z(\theta_z)$. Moreover, the ζ-policy *achieves an indirect utility of*

$$\mathbb{E}_s\left[\sum_z \beta^{t(z)}\zeta c_z^A(\theta)\right] = \zeta\mathbb{E}_s\left[\sum_z \beta^{t(z)} c_z^A(\theta)\right] = \zeta V^A(\theta, n_s^A; s) = \frac{n_s^B}{n_s^A} V^A(\theta, n_s^A; s)$$

Thus, for a given productivity θ_s in a given state, s, the ratio $\frac{V(\theta, n_s; s)}{n_s}$ is constant. So

$$V(\theta, n_s; s) = a_s(\theta_s) n_s$$

and we conclude that the history of idiosyncratic shocks is irrelevant up to the current value of net-worth. So it makes sense to define the value function as a function of only contemporaneous productivity and contemporaneous net-worth

$$V(\theta_s, n_s; s) = a_s(\theta_s) n_s$$

It follows, also, that the optimal policies are linear in net-worth, as the ζ-policy for agent B achieves an indirect utility of $V^B(\theta_s, n_s^B; s)$. Moreover, the optimal policies per unit of net-worth will be a function only of current productivity and independent of the history of productivity shocks– previous productivity shocks influence the value of net-worth and the optimal policies are linear in net-worth.

To derive the properties of the coefficient $a_s(\theta_s)$ and the optimal consumption rule I study the entrepreneur's decision problem using the value function, i.e.,

$$\begin{gathered} \max\left\{c_s(\theta_s) + \beta\mathbb{E}_s\left[V(\theta_{s'}, n_{s'}(\theta_s); s')\right]\right\} \\ \textit{s.t. equations (3), (4), (6) and (7)} \end{gathered} \tag{11}$$

Note that by the Law of Total Expectations: $\mathbb{E}_s\left[V(\theta_{s'}, n_{s'}(\theta_s); s')\right] = \sum_{s'} \mathbb{P}(s'|s)\mathbb{E}_{s'}\left[V(\theta_{s'}, n_{s'}(\theta_s); s')\right]$, then the FOC are given by

$$\begin{aligned}
&(c_s) && 0 = 1 - \lambda_s(\theta_s) + \eta_{c,s}(\theta_s) \\
&(k_s) && 0 = -\lambda_s(\theta_s) p_s + \mu_s(\theta_s) + \eta_{k,s}(\theta_s) + \sum_{s'} \tilde{\lambda}_{s'}(\theta_s)[\theta_s + (1-\delta)p_{s'}] \\
&(b_{\phi,s}) && 0 = -\lambda_s(\theta_s) q_{\phi,s} - \mu_s(\theta_s)\partial_0^{[-1,0]}(b_{\phi,s}) + \sum_{s'} \tilde{\lambda}_{s'}(\theta_s) d(\phi, s') \\
&(n'_{s'}) && 0 = \beta\mathbb{P}(s'|s)\mathbb{E}_{s'}[V_n(\theta_{s'}, n_{s'}(\theta_s); s')] - \tilde{\lambda}_{s'}(\theta_s)
\end{aligned}$$

where $\partial_0^{[-1,0]}(b)$ is the subdifferential of the negative-part function, i.e.,

$$\partial_0^{[-1,0]}(b) = \begin{cases} -1 & \textit{if } b < 0 \\ \in [-1, 0] & \textit{if } b = 0 \\ 0 & \textit{if } b > 0 \end{cases}$$

Let $R^*(\theta_s; s')$ be the random return per unit invested on the optimal portfolio by entrepreneur θ_s.

Next I derive the relationship between the after-capital-investment-marginal value of net-worth $\lambda_s(\theta_s)$ and the optimal return $R^*(\theta_s; s')$. The optimality of the portfolio imply the following.[6]

$$\left(b^*_{\phi,s}(\theta_s)\right)_+ \left[\lambda_s(\theta_s) q_{\phi,s} - \sum_{s'} \tilde{\lambda}_{s'}(\theta_s) d(\phi, s')\right] = 0$$

$$\left(b^*_{\phi,s}(\theta_s)\right)_- \left[\lambda_s(\theta_s) q_{\phi,s} - \mu_s(\theta_s) - \sum_{s'} \tilde{\lambda}_{s'}(\theta_s) d(\phi, s')\right] = 0$$

$$k^*_{k,s}(\theta_s) \left[\lambda_s(\theta_s) p_s - \mu_s(\theta_s) - \sum_{s'} \tilde{\lambda}_{s'}(\theta_s)[\theta_s + (1-\delta) p_{s'}]\right] = 0$$

Using that $b = b_+ - b_-$ we get that

$$0 = p_s k^*_s(\theta_s) \left[\lambda_s(\theta_s) - \sum_{s'} \tilde{\lambda}_{s'}(\theta_{s'}) \frac{\theta_s + (1-\delta) p_{s'}}{p_s}\right] + \sum_{\phi \in \Phi} q_{\phi,s} b^*_{\phi,s}(\theta_s) \left[\lambda_s(\theta_s) - \sum_{s'} \tilde{\lambda}_{s'}(\theta_{s'}) \frac{d(\phi, s')}{q_{\phi,s}}\right] - \mu_s(\theta_s) \left[k^*_s(\theta_s) - \sum_{\phi \in \Phi} \left(b^*_{\varphi,s}(\theta_s)\right)_-\right] \quad (12)$$

But the last term is zero by the complementary slackness condition for the collateral constraint inequality. Then, if savings are positive

$$R^*(\theta_s; s') = \frac{\theta_s + (1-\delta) p_{s'}}{p_s} \frac{p_s k^*_s(\theta_s)}{p_s k^*_s(\theta_s) + \sum_{\varphi \in \Phi} q_{\varphi,s} b^*_{\varphi,s}(\theta_s)} + \sum_{\phi \in \Phi} \frac{d(\phi, s')}{q_{\phi,s}} \frac{q_{\phi,s} b^*_{\phi,s}(\theta_s)}{p_s k^*_s(\theta_s) + \sum_{\varphi \in \Phi} q_{\varphi,s} b^*_{\varphi,s}(\theta_s)}$$

Thus,

$$\lambda_s(\theta_s) = \sum_{s'} \tilde{\lambda}_{s'}(\theta) R^*(\theta_s; s') = \beta \sum_{s'} \mathbb{P}(s'|s) \mathbb{E}_{s'}[V_n(\theta_{s'}, n_{s'}(\theta_s); s')] R^*(\theta_s; s') = \beta \mathbb{E}_s[a_{s'}(\theta_{s'}) R^*(\theta_s; s')]$$

Alternatively, when savings are zero we have that consumption is positive, so $\lambda_s(\theta_s) = 1$. In this case it is still possible to define the optimal portfolio as the solution to the following portfolio problem, where the collateral constraint and the capital no-short-sale constraint have been recasted in terms of portfolio weights.

[6] Note that the $(b^*_{\phi,s}(\theta))_+$-equation is the relevant equation for a lender, whereas the $(b^*_{\phi,s}(\theta))_-$-equation is the relevant equation for a borrower.

$$\max_{\omega_{j,s(\theta_s)}} \mathbb{E}_s \left[a_{s'}(\theta_{s'}) \sum_{j \in \{k\} \cup \Phi} \omega_{j,s} R(\theta_s; s') \right] = \max_{\omega_{j,s(\theta_s)}} \mathbb{E}_s \left[a_{s'}(\theta_{s'}) R^*(\theta_s; s') \right] \tag{13}$$

$$\text{s.t.} \quad \sum_{j \in \{k\} \cup \Phi} \omega_{j,s} = 1$$

$$\sum_{\phi \in \Phi} \left(\frac{\omega_{\phi,s}}{q_{\phi,s}} \right)_- \leq \frac{\omega_{k,s}}{p_s}$$

$$\omega_{k,s} \geq 0$$

When total savings are zero, the expected discounted value of the optimal portfolio return cannot be greater than the current marginal value of wealth, i.e.,

$$\lambda_s(\theta_s) = 1 \geq \beta \mathbb{E}_s[a_{s'}(\theta_{s'}) R^*(\theta_s; s')]$$

with equality when the collateral and capital no-short contraints are slack. So we have that

$$\lambda_s(\theta_s) = \max \{1, \beta \mathbb{E}_s[a_{s'}(\theta_{s'}) R^*(\theta_s; s')]\}$$

On the other hand, by the Envelope Theorem we have that

$$V_n(\theta_s, n_s; s) = a_s(\theta_s) = \lambda_s(\theta_s) \left\{ 1 + p_s \Psi'^{-1}(p_s) - \Psi\left(\Psi'^{-1}(p_s) \right) \right\} > 0 \tag{14}$$

where the inequality follows from the fact that $\lambda_s(\theta_s) \geq 1$ *and* $p_s \Psi'^{-1}(p_s) - \Psi(\Psi'^{-1}(p_s)) > 0$ *since the net-value of capital production is non-negative. Therefore,*

$$a_s(\theta_s) = \left\{ 1 + p_s \Psi'^{-1}(p_s) - \Psi\left(\Psi'^{-1}(p_s) \right) \right\} \max \{1, \beta \mathbb{E}_s[a_{s'}(\theta_{s'}) R^*(\theta_s; s')]\}$$

From where it follows that $a_s(\theta_s)$ *is a non-decreasing function of* θ_s*. In fact, if* $\theta_s^A > \theta_s^B$*, then* $R^*(\theta_s^A; s') \geq R^*(\theta_s^B; s')$*, as the more productive entrepreneur* θ_s^A *can always allocate the fractions of savings to each asset as done by the less productive entrepreneur* θ_s^B*; and conditional on the future aggregate state* s'*, the expected marginal value of net worth* $\mathbb{E}_{s'}[a_{s'}(\theta_{s'})]$ *is the same for both agents.*

Finally, the optimal consumption policy mus satisfy that $c_s(\theta_s) = 0$ *if* $\lambda_s(\theta_s) > 1$*, which is equivalent to* $\beta \mathbb{E}_s \left[a_{s'}(\theta_{s'}) R^*(\theta_s; s') \right] > 1$. ■

This completes the characterization of the entrepreneurs' decision problem. Now I turn to the characterization of the equilibrium.

4 Equilibrium Characterization

In order to characterize the unique equilibrium of this economy I proceed in two steps. Fisrt, I show that in equilibrium entrepreneurs sort in two groups according to their idyosincratic productivities: productive entrepreneurs and lenders. Second, I show that there exists state prices that can be used to price *all* the assets in the economy–capital and debt contracts. In this setting, with credit constraints the existence of state prices is not guaranteed by no arbitrage. Intuitively, different agents may have different valuations in the different future states, and different groups of agents may be pricing different assets, so no single set of state prices prices *all* assets. But in the present setting the relative valuations of present and expected future wealth only depends on the current productivity, as future productivities are i.i.d, thus *all* lenders and the marginal buyer of capital, who is indifferent between lending and buying capital, will value future wealth and assets identically in equilibrium. Thus it will possible to establish the existence of state prices, which is done in Propositions 2 and 3 that at the same time characterize the optimal portfolio for each entrepreneur.

In order to characterize equilibrium asset prices, first I state the following result that characterize the set of landlords.

The following proposition shows that entrepreneurs with productivities above a threshold $\hat{\theta}_s$ will hold capital.

Proposition 1 (The Set of Productive Entrepreneurs) *Consider two entrepreneurs with productivities θ_s^A and θ_s^B in state s, s.t. $\theta_s^A > \theta_s^B$, and that θ_s^B holds land in state s, then θ_s^A will hold land in state s, i.e., $k_s^{*,B}(\theta_s^B) > 0$ then $k_s^{*,A}(\theta_s^A) > 0$. Therefore, in equilibrium exists $\hat{\theta}_s < \infty$ such that an entrepreneur $\theta_s > \hat{\theta}_s$ will hold capital and produce. Formally, $\hat{\theta}_s$ is the infimum productivity of the set of productive entrepreneurs, and the difference between the set of productive entrepreneurs and $[\hat{\theta}_s, \bar{\theta}]$ has $dF(\theta)$-measure zero.*

Proof of Proposition 1: *I proove the first part by contradiction. Suppose the more productive entrepreneur does not hold land, i.e., $k_s^{*,A}(\theta_s^A) = 0$. Then, it must be that for any other policy with the same consumption in state s*

$$\mathbb{E}_s\left[V(\theta_{s'}, n_{s'}^{*,A}(\theta_s^A); s')\right] \geq \mathbb{E}_s\left[V(\theta_{s'}, n_{s'}^{A}(\theta_s^A); s')\right]$$

and using Lemma 1

$$\mathbb{E}_s\left[a_{s'}(\theta_{s'})R^*(\theta_s^A; s')\right] \geq \mathbb{E}_s\left[a_{s'}(\theta_{s'})R(\theta_s^A; s')\right]$$

Let me introduce the following notation for the outcomes that agent θ_s^A could obtain using the same portfolio weights as agent θ_s^B. Let $R(\theta_s^A; \theta_s^B, s')$ be the portfolio returns obtained by θ_s^A allocating *the same fractions of savings to each asset as agent* θ_s^B.

$$R(\theta_s^A;\theta_s^B,s') = \frac{1}{p_s k_s^{*,B}(\theta_s) + \sum_{\varphi\in\Phi} q_{\varphi,s} b_{\varphi,s}^{*,B}(\theta_s^B)} \left\{ \frac{\theta_s^A + (1-\delta)p_{s'}}{p_s} p_s k_s^{*,B}(\theta_s^B) + \sum_{\phi\in\Phi} \frac{d(\phi,s')}{q_{\phi,s}} q_{\phi,s} b_{\phi,s}^{*,B}(\theta_s) \right\}$$

Then, from above it must be that

$$\mathbb{E}_s\left[a_{s'}(\theta_{s'})R^*(\theta_s^A;s')\right] \geq \mathbb{E}_s\left[a_{s'}(\theta_{s'})R(\theta_s^A;\theta_s^B,s')\right]$$

On the other hand, optimality of the choices by agent θ_s^B *imply that, if the less productive entrepreneur* θ_s^B *would choose her portfolio using the same portfolio weights as agent* θ_s^A*, she will not be able to get a better expected value of wealth, so it must be that*

$$\mathbb{E}_s\left[a_{s'}(\theta_{s'})R^*(\theta_s^B;s')\right] \geq \mathbb{E}_s\left[a_{s'}(\theta_{s'})R(\theta_s^B;\theta_s^A,s')\right]$$

But, since the more productive entrepreneur does not hold land, $k_s^{*,A}(\theta_s^A) = 0$*, then* $R(\theta_s^B;\theta_s^A,s') = R^*(\theta_s^A;s')$*; and since the least productive entrepreneur* θ_s^B *does hold land,* $k_s^{*,B}(\theta_s^B) > 0$*, and,* $\theta_s^A > \theta_s^B$*, then* $R(\theta_s^A;\theta_s^B,s') > R^*(\theta_s^B;s')$*. Then, as* $\mathbb{E}_s\left[a_{s'}(\theta_{s'})R(s')\right]$ *is an increasing function in* $R(s')$*, we get that*

$$\mathbb{E}_s\left[a_{s'}(\theta_{s'})R(\theta_s^A;\theta_s^B,s')\right] > \mathbb{E}_s\left[a_{s'}(\theta_{s'})R^*(\theta_s^A;s')\right] \qquad \rightarrow\!\leftarrow$$

To proove the second part for any state s, let $\mathcal{K}_s$ *be the set of productivities of productive entrepreneurs, i.e.* $\mathcal{K}_s = \{\vartheta_s : k_s^*(\vartheta_s) > 0\}$*. Now let*

$$\hat{\theta}_s = \inf\{\vartheta \in \mathcal{K}_s\}$$

Note that in equilibrium this set must be nonempty, so $\hat{\theta}_s < \infty$*. From above for any* $\theta_s > \hat{\theta}_s$*,* $k_s^*(\theta_s) > 0$*. In fact, this is trivially the case if* $\hat{\theta}_s \in \mathcal{K}_s$*. If* $\hat{\theta}_s \notin \mathcal{K}_s$*, then there must exist* $\vartheta_s \in \mathcal{K}_s$ *such that* $\vartheta_s < \theta_s$*, otherwise* θ_s *would be the infimum which is a contradiction. So we conclude that the set of productive entrepreneurs, in state s, is a subset of* $[\hat{\theta}_s,\bar{\theta}]$*, and thus, the difference with* $]\hat{\theta}_s,\bar{\theta}]$ *is a* $dF(\theta)$*-measure zero set.* ■

An imediate consequence of Proposition 1 is that debt contracts, if traded, will be held by entrepreneurs $\theta_s < \hat{\theta}_s$ that I will refer to as lenders.

The following two propositions show the existence of state prices in this economy with credit frictions, characterize the price of capital and debt contracts, and characterize the optimal portfolio of each entrepreneur.

Proposition 2 (State and Debt-Contracts Prices) *Under the maintained assumptions, for any state s, there exist state prices give by*

$$\psi_{s'} = \beta\mathbb{P}(s'|s)\mathbb{E}_{s'}\left[a_{s'}(\theta_{s'})\right] \tag{15}$$

such that

$$q_{\phi,s} \geq \psi_{sU} \min\{\phi, (1-\delta)p_{sU}\} + \psi_{sD} \min\{\phi, (1-\delta)p_{sD}\} \tag{16}$$

with equality for debt contracts traded in equilibrium.

Proof of Proposition 2: *Since the marginal value of net-worth $a_s(\theta_s)$ is a non-decreasing function of θ_s, so it is the after-capital-investment-marginal value of net-worth $\lambda_s(\theta_s)$. Therefore, in equilibrium, it must be that $\lambda_s(\theta_s) = 1$ for all lenders. In fact, suppose $\lambda_s(\theta_s) > 1$ for some lenders. For any $\theta_s < \hat{\theta}_s$, from Proposition 1 $k^*_s(\theta_s) = 0$, and from the collateral constraint $b^*_{\phi,s}(\theta_s) \geq 0$. Thus the feasible returns for lenders are a linear combination of the return on debt contracts $\frac{d(\phi;s')}{q_{\phi,s}}$. But these returns are independent of entrepreneur's productivity, so it must be that the optimal return for all lenders is the same. Then, if for some lenders $\lambda_s(\theta_s) > 1$ with $\lambda_s(\theta_s) = \max\{1, \beta\mathbb{E}_s[a_{s'}(\theta_{s'})R^*(\theta_s; s')]\}$, it must be that for all lenders $\lambda_s(\theta_s) > 1$. Therefore, aggregate demand for the consumption good is zero, which is inconsistent with market clearing for current consumption.*

Then it follows that

$$1 \geq \beta\mathbb{E}_s[a_{s'}(\theta_{s'})R^*(\theta_s; s')]$$

and then for any debt contract ϕ

$$1 \geq \beta\mathbb{E}_s\left[a_{s'}(\theta_{s'})\frac{d(\phi; s')}{q_{\phi,s}}\right]$$
$$q_{\phi,s} \geq \psi_{sU} \min\{\phi, (1-\delta)p_{sU}\} + \psi_{sD} \min\{\phi, (1-\delta)p_{sD}\}$$

If some debt contract ϕ is traded in equilibrium, then some lender, $\theta_s < \hat{\theta}_s$, must purchase this contract in which case his savings are positive, so

$$\lambda_s(\theta_s) = 1 = \beta\mathbb{E}_s\left[a_{s'}(\theta_{s'})R^*(\theta_s; s')\right] = \beta\mathbb{E}_s\left[a_{s'}(\theta_{s'})\frac{d(\phi; s')}{q_{\phi,s}}\right]$$

where the last equality follows from the fact that if debt contract ϕ is traded, the lender needds to be getting a discounted expected return of at least 1. Then

$$q_{\phi,s} = \psi_{sU} \min\{\phi, (1-\delta)p_{sU}\} + \psi_{sD} \min\{\phi, (1-\delta)p_{sD}\}$$

■

Now I turn to the funding decision of productive entrepreneurs and show that the price of capital can also be computed using the state prices.

Proposition 3 (Leverage and the Price of Capital) *Capital is priced by the marginal entrepreneur $\hat{\theta}_s$, i.e.,*

$$p_s = \psi_{sU}\left[\hat{\theta}_s + (1-\delta)p_{sU}\right] + \psi_{sD}\left[\hat{\theta}_s + (1-\delta)p_{sD}\right] \tag{17}$$

And any entrepreneur that is strictly more productive than the marginal entrepreneur will buy land *using the maximum leverage, i.e., it will pledge all her capital as collateral and will issue the debt contract that promise the maximum future value of capital.*

Proof of Proposition 3: *For this proof first note that since the after-capital-investment-marginal-value of net-worth, $\lambda_s(\theta_s) \geq 1$ it must be that, for a productive entrepreneur, the optimal return for a portfolio that invest in capital satisfies*

$$1 \leq \beta \mathbb{E}_s \left[a_{s'}(\theta_{s'}) R^*(\theta_s; s') \right]$$

Then

$$\begin{aligned} 1 \leq & \frac{\beta}{p_s k_s^*(\theta_s) + \sum_{\varphi \in \Phi} q_{\varphi,s} b_{\varphi,s}^*(\theta_s)} \\ & \qquad \times \mathbb{E}_s \left[a_{s'}(\theta_{s'}) \left\{ \frac{\theta_s + (1-\delta) p_{s'}}{p_s} p_s k_s^*(\theta_s) + \sum_{\phi \in \Phi} \frac{d(\phi; s')}{q_{\phi,s}} q_{\phi,s} b_{\phi,s}^*(\theta_s) \right\} \right] \\ \leq & \frac{1}{p_s k_s^*(\theta_s) + \sum_{\varphi \in \Phi} q_{\varphi,s} b_{\varphi,s}^*(\theta_s)} \left[p_s k_s^*(\theta_s) \left\{ \psi_{sU} \frac{\theta_s + (1-\delta) p_{sU}}{p_s} + \psi_{sD} \frac{\theta_s + (1-\delta) p_{sD}}{p_s} \right\} \right. \\ & \qquad \left. + \sum_{\phi \in \Phi} q_{\phi,s} b_{\phi,s}^*(\theta_s) \frac{\psi_{sU} d(\phi; sU) + \psi_{sD} d(\phi; sD)}{q_{\phi,s}} \right] \end{aligned}$$

$$\begin{aligned} & p_s k_s^*(\theta_s) + \sum_{\varphi \in \Phi} q_{\varphi,s} b_{\varphi,s}^*(\theta_s) \leq \\ & \qquad p_s k_s^*(\theta_s) \left\{ \psi_{sU} \frac{\theta_s + (1-\delta) p_{sU}}{p_s} + \psi_{sD} \frac{\theta_s + (1-\delta) p_{sD}}{p_s} \right\} + \sum_{\phi \in \Phi} q_{\phi,s} b_{\phi,s}^*(\theta_s) \end{aligned}$$

$$p_s \leq \psi_{sU} \left[\theta_s + (1-\delta) p_{sU} \right] + \psi_{sD} \left[\theta_s + (1-\delta) p_{sD} \right] \tag{18}$$

Thus

$$p_s = \psi_{sU} \left[\hat{\theta}_s + (1-\delta) p_{sU} \right] + \psi_{sD} \left[\hat{\theta}_s + (1-\delta) p_{sD} \right]$$

It follows from equation (18) that $p_s \leq \psi_{sU} \left[\hat{\theta}_s + (1-\delta) p_{sU} \right] + \psi_{sD} \left[\hat{\theta}_s + (1-\delta) p_{sD} \right]$. Now suppose $p_s < \psi_{sU} \left[\hat{\theta}_s + (1-\delta) p_{sU} \right] + \psi_{sD} \left[\hat{\theta}_s + (1-\delta) p_{sD} \right]$. Let $\varepsilon > 0$ be an arbitrary small number, then for another entrepreneur $\vartheta_s = \hat{\theta}_s - \varepsilon$ it will be the case that $p_s < \psi_{sU} \left[\vartheta_s + (1-\delta) p_{sU} \right] + \psi_{sD} \left[\vartheta_s + (1-\delta) p_{sD} \right]$. But this imply that by holding capital ϑ_s can get a marginal value of net-worth after-capital-investment strictly greater than 1, which is a contradiction as $\vartheta_s < \hat{\theta}_s$.

To show that capitalists (entrepreneurs $\theta_s > \hat{\theta}_s$) use maximum leverage, note that the after-capital-investment marginal value of net-worth, $\lambda_s(\theta_s) > 1$ and their valuation of debt contracts equals the market value of these contracts. It follows from above that for any entrepreneur $\theta_s > \hat{\theta}_s$, $p_s < \psi_{sU} \left[\theta_s + (1-\delta) p_{sU} \right] + \psi_{sD} \left[\theta_s + (1-\delta) p_{sD} \right]$, so the after-capital-investment marginal value of net-worth, $\lambda_s(\theta_s) > 1$. In addition, state prices are independent of entrepreneurs current *productivity, given that productivity shocks are i.i.d. Thus entrepreneurs $\theta_s > \hat{\theta}_s$ do not buy debt contracts–i.e., lend–and if they sell these contracts–i.e., borrow–they pledge all their capital as collateral. In fact, if for some debt contract $b^*_{\phi,s}(\theta_s) > 0$, then from the FOC we have that*

$$\lambda_s(\theta_s) = \tilde{\lambda}_{s'}(\theta_s)\frac{d(\phi; s')}{q_{\phi,s}} = \frac{\psi_{sU}d(\phi; sU) + \psi_{sD}d(\phi; sD)}{q_{\phi,s}} = 1$$

On the other hand, if entrepreneur $\theta_s > \hat{\theta}_s$ borrows, from the FOC we have that

$$\mu_s(\theta_s) = \lambda_s(\theta_s)q_{\phi,s} - \tilde{\lambda}_s(\theta_s)d(\phi; s') = \lambda_s q_{\phi,s} - \psi_{sU}d(\phi; sU) - \psi_{sD}d(\phi; sD) > 0$$

so we conclude that for entrepreneurs that borrow $\sum_\phi \big(b^*_{\phi,s}(\theta_s)\big)_- = k^*_s(\theta_s)$, i.e., they pledge all their capital as collateral.

Note that it suffice to consider that entrepreneurs borrow using a single arbitrary contract. To show that this is the case, below, I show that: (i) any pair of debt contracts $\phi_1, \phi_2 \in [(1 - \delta)\min_{s'}\{p_{s'}\}, (1-\delta)\max_{s'}\{p_{s'}\}]$, can be replicated by a single contract, $\phi \in [(1-\delta)\min_{s'}\{p_{s'}\}, (1 - \delta)\max_{s'}\{p_{s'}\}]$, such that, using the same amount of collateral, the entrepreneur can obtain the same amount of credit in state s, and will have to make the same effective deliveries in future states s'; and (ii) any debt contract $\varphi < (1-\delta)\min_{s'}\{p_{s'}\}$ can be replicated by debt contract $(1-\delta)\min_{s'}\{p_{s'}\}$ using strictly less collateral. To show (i) consider the case where an entrepreneur borrows using two debt contracts $\phi_1, \phi_2 \in [(1 - \delta)\min_{s'}\{p_{s'}\}, (1 - \delta)\max_{s'}\{p_{s'}\}]$. Let $b_{\phi_1,s}, b_{\phi_2,s}$ be (minus) the number of contracts issued by the entrepreneur. Let $\xi = b_{\phi_1,s}/(b_{\phi_1,s} + b_{\phi_2,s})$ be the fraction of debt contracts ϕ_1 being issued. Then, $\xi \in [0, 1]$. Consider the following replicating portfolio, let $\phi = \xi\phi_1 + (1-\xi)\phi_2$ and $b_{\phi,s} = b_{\phi_1,s} + b_{\phi_2,s}$. Since $\xi \in [0, 1]$, $\phi \in [(1-\delta)\min_{s'}\{p_{s'}\}, (1-\delta)\max_{s'}\{p_{s'}\}]$, and since $b_{\phi,s}$ equals (minus) the number of contracts being issued it is feasible. Since $\phi, \phi_1, \phi_2 \in [(1 - \delta)\min_{s'}\{p_{s'}\}, (1 - \delta)\max_{s'}\{p_{s'}\}]$

$$\begin{gathered}\xi d(\phi_1; s') + (1 - \xi)d(\phi_2; s') = d(\phi; s') \\ b_{phi_1,s}d(\phi_1; s') + b_{\phi_2,}d(\phi_2; s') = (b_{\phi_1,s} + b_{\phi_2,s})\left(\xi d(\phi_1; s') + (1 - \xi)d(\phi_2; s')\right) = b_{\phi,s}d(\phi; s')\end{gathered}$$

That is the replicating portfolio does the same effective deliveries as the original one. Moreover, (minus) the amount borrowed is also equal to the original portfolio:

$$\begin{aligned}q_{\phi_1,s}b_{\phi_1,s} + q_{\phi_2,s}b_{\phi_2,s} =& \psi_{sU}(b_{\phi_1,s} + b_{\phi_2,s})\left(\xi d(\phi_1, sU) + (1 - \xi)d(\phi_2, sU)\right) \\ &+ \psi_{sD}(b_{\phi_1,s} + b_{\phi_2,s})\left(\xi d(\phi_1, sD) + (1 - \xi)d(\phi_2, sD)\right) \\ =& (b_{\phi_1,s} + b_{\phi_2,s})\left(\psi_{sU}d(\phi, sU) + \psi_{sD}d(\phi, sD)\right) = b_{\phi,s}q_{\phi,s}\end{aligned}$$

To prove (ii) consider the case of an entrepreneur borrowing using debt contract $\varphi < \min_{s'}\{(1-\delta)p_{s'}\}$, and let $b_{\varphi,s}$ be (minus) the number of contracts issued by the entrepreneur. Let $\xi = \varphi/((1 - \delta)\min_{s'}\{p_{s'}\}) < 1$, and consider the following replicating portfolio. Let $\phi = (1 - \delta)\min_{s'}\{p_{s'}\}$ and $b_{\phi,s} = \xi b_{\varphi,s}$ be (minus) the number of ϕ contracts issued. The effective deliveries of both portfolios are

$$b_{\varphi,s}d(\varphi; s') = b_{\varphi,s}\varphi = b_{\varphi,s}\xi(1 - \delta)\min_{s'}\{p_{s'}\} = b_{\phi,s}\phi = b_{\phi,s}d(\phi; s')$$

Moreover, the amount borrowed using strictly less collateral is given by

$$q_{\varphi,s}b_{\varphi,s} = q_{\phi,s}\xi b_{\varphi,s} = q_{\phi,s}b_{\phi,s}$$

where I used that if contract ϕ where to trade in equilibrium it will do so at price $q_{\phi,s} = \phi(\psi_{sU} + \psi_{sD})$ (Proposition 2).

Finally, I want to show that the after-capital-investment marginal value of net-worth, $\lambda_s(\theta_s)$ is a strictly increasing function of the debt contract used ϕ, if $\phi < (1-\delta)\max_{s'}\{p_{s'}\}$. From Proposition 2 and the expression for the price of capital just proved, we have that

$$p_s - q_{\phi,s} = \psi_{sU}\left[\hat{\theta}_s + ((1-\delta)p_{sU} - \phi)_+\right] + \psi_{sD}\left[\hat{\theta}_s + ((1-\delta)p_{sD} - \phi)_+\right]$$

On the other hand, the after-capital-investment marginal value of wealth in state s of investing in capital borrowing using contract ϕ, and pledging all capital as collateral, for agent θ_s is given by,

$$\begin{aligned}\lambda_s(\theta_s) &= \frac{\psi_{sU}\left[\theta_s + ((1-\delta)p_{sU} - \phi)_+\right] + \psi_{sD}\left[\theta_s + ((1-\delta)p_{sD} - \phi)_+\right]}{p_s - q_{\phi,s}} \\ &= \frac{\psi_{sU}\left[\theta_s + ((1-\delta)p_{sU} - \phi)_+\right] + \psi_{sD}\left[\theta_s + ((1-\delta)p_{sD} - \phi)_+\right]}{\psi_{sU}\left[\hat{\theta}_s + ((1-\delta)p_{sU} - \phi)_+\right] + \psi_{sD}\left[\hat{\theta}_s + ((1-\delta)p_{sD} - \phi)_+\right]}\end{aligned}$$

An entrepreneur chooses the debt contract to issue to maximize the after-capital-investment marginal value of wealth in state s, $\lambda_s(\theta_s)$. Taking derivative wrt ϕ in the previous expression we get, [7]

$$\begin{aligned}\frac{\partial\lambda_s(\theta_s)}{\partial\phi} =& \frac{-\psi_{sU}\partial_0^{[0,1]}\left((1-\delta)p_{sU} - \phi\right) - \psi_{sD}\partial_0^{[0,1]}\left((1-\delta)p_{sD} - \phi\right)}{\psi_{sU}\left[\hat{\theta}_s + ((1-\delta)p_{sU} - \phi)_+\right] + \psi_{sD}\left[\hat{\theta}_s + ((1-\delta)p_{sD} - \phi)_+\right]} \\ &- \frac{\left(\psi_{sU}\left[\theta_s + ((1-\delta)p_{sU} - \phi)_+\right] + \psi_{sD}\left[\theta_s + ((1-\delta)p_{sD} - \phi)_+\right]\right)}{\left(\psi_{sU}\left[\hat{\theta}_s + ((1-\delta)p_{sU} - \phi)_+\right] + \psi_{sD}\left[\hat{\theta}_s + ((1-\delta)p_{sD} - \phi)_+\right]\right)^2} \\ &\times\left(\psi_{sU}\partial_0^{[0,1]}\left((1-\delta)p_{sU} - \phi\right) + \psi_{sD}\partial_0^{[0,1]}\left((1-\delta)p_{sD} - \phi\right)\right)\end{aligned}$$

Since $\phi < (1-\delta)\max_{s'}\{p_{s'}\}$, $\max\{\partial_0^{[0,1]}\left((1-\delta)p_{s'} - \phi\right)\} > 0$ *and* $\max\{((1-\delta)p_{s'} - \phi)_+\} > 0$*, then*

$$\frac{d\lambda_s(\theta_s)}{d\phi} > 0$$

$$\begin{aligned}&\psi_{sU}\left[\hat{\theta}_s + ((1-\delta)p_{sU} - \phi)_+\right] + \psi_{sD}\left[\hat{\theta}_s + ((1-\delta)p_{sD} - \phi)_+\right] \\ &\qquad > \psi_{sU}\left[\theta_s + ((1-\delta)p_{sU} - \phi)_+\right] + \psi_{sD}\left[\theta_s + ((1-\delta)p_{sD} - \phi)_+\right]\end{aligned}$$

$$\hat{\theta}_s > \theta_s$$

So capitalists, $\theta_s > \hat{\theta}_s$ can be characterized as borrowing using a single debt contract and pledging

[7] Recall that $\partial_{x_0}^{[a,b]}(x)$ denotes the correspondence taking values: a if $x < x_0$; $[a, b]$ if $x = x_0$; and b if $x > x_0$.

all their capital as collateral, and they will issue the debt contract that promises the largest value of capital, i.e., $\hat{\phi}_s = (1-\delta)\max_{s'}\{p_{s'}\}$. *As I just showed that if, in equilibrium, they were to issue a contract with a strictly smaller promise they can increase their after-capital-investment marginal value of wealth in state s,* $\lambda_s(\theta_s)$*, and therefore be better off, by increasing the amount promised. That is, capitalists use the maximum leverage, which can be measured as the loan-to-value,* $LTV_s = q_{\hat{\phi}_s,s}/p_s$. ■

Proposition 3 establishes that entrepreneur in this economy will issue risky debt, a result that is in stark contrast with Fostel and Geanakoplos (2012) No-Default Theorem. To gain further intuition on the optimality of risky debt in my model it is useful to compare both results. The key behind Fostel and Geanakoplos (FS) result is that an agent issuing a debt-contract $\varphi \in (\min_{s'}\{p_{s'}\}, \max_{s'}\{p_{s'}\}]$ can obtain the same payoffs at the same cost, issuing only the max-min debt contract, $\phi_{mm} = (1-\delta)\min_{s'}\{p_{s'}\}$. In contrast in my setting a productive entrepreneur is better of by using debt-contract $\hat{\phi}_s = (1-\delta)\max_{s'}\{p_{s'}\}$. The reason is that a more productive entrepreneur by issuing the largest promise is maximizing the leverage in its capital purchase, and this will yield the maximum amount of production in the next period. As this entrepreneur is more productive than the productivity implied by the price of capital, he is effectively buying future consumption at below market prices. In contrast in FS setting there is no inter-temporal production, so there are no benefits of increasing the leverage above $q_{\phi_{mm},s}/p_s$, at equilibrium prices. To sum up, it is the heterogeneity in the productivity of inter-temporal production that makes risky debt optimal in equilibrium.

Aggregation. In this environment with heterogeneous entrepreneurs, in general, equilibrium prices will depend on the joint distribution of idiosyncratic productivities and net-worth. Though, in this setting with i.i.d. productivity shocks and optimal policies that are linear in net-worth, aggregate demand for assets will depend only on average net-worth. This greatly simplifies the problem of solving for equilibrium, by reducing the dimensionality of the entrepreneurs decision problem. To illustrate I show how the aggregate demand for capital can be computed only from average entrepreneurial net-worth.[8] For this let me introduce the following notation. Let $\bar{n}_s$ be the average net-worth at the beginning of the period in state s. Since net-worth at the beginning of the period only depends on previous productivity shocks,

$$\bar{n}_s = \int_{\Theta^*} n_s(\theta_{s^*})\, dF_{s^*}(\theta_{s^*})$$

Then, since productivity shocks are independent, we can express the aggreate demand for capital

[8] In Appendix A the interested reader can find the expressions for the aggregate demand and supply of capital and debt contract $\hat{\phi}_s$.

as

$$D_k(p_s, q_{\hat{\phi}_s,s}) = \int_{\Theta^*} \int_{\hat{\theta}_s}^{\bar{\theta}} \frac{n_s(\theta_{s^*})}{p_s - q_{\hat{\phi}_s,s}} \, dF_s(\theta) dF_{s^*}(\theta_{s^*}) = \left(1 - F(\hat{\theta}_s)\right) \frac{\bar{n}_s}{p_s - q_{\hat{\phi}_s,s}}$$

Now that the equilibrium has been fully characterized I turn to how I solve for equilibrium.

5 Solving for Equilibrium

I have showed that in equilibrium, depending on their current productivity θ_s, entrepreneurs will sort into: (*i*) *capitalists*, $\theta_s > \hat{\theta}_s$, who buy and produce using capital and pledge all their capital to borrow using debt contract $\hat{\phi}_s = (1 - \delta) \max\{p_{s'}\}$; and (*ii*) *lenders*, $\theta_s \leq \hat{\theta}_s$, who sell their capital and lend buying debt-contract $\hat{\phi}_s$ (Figure 2). In addition, I established the existence of state prices (equation 15) and the irrelevance of the distribution of net-worth. It follows from the characterization of the equilibrium that the identity of the marginal entrepreneur is given by

$$\hat{\theta}_s = \frac{p_s - q_{\hat{\phi}_s,s}}{\psi_{sU} + \psi_{sD}} \tag{19}$$

This expression is obtained from the indifference condition for the marginal entrepreneur between lending and buying capital using maximum leverage.[9]

In the characterization of the equilibrium I specified the optimal allocations for every entrepreneur given their current productivity θ_s, thus to solve for equilibrium I just need to find a set of prices for capital and the debt-contract $\hat{\phi}_s$, $\left\{p_s, q_{\hat{\phi}_s,s}\right\}_{s=0,U,D}$, such that markets clear in every state $s = 0, U, D$.

Note that given that lenders are indifferent between lending and consuming, in each state, the demand for debt contracts is undetermined.[10] In equilibrium the demand will meet the supply of these debt contracts, but it is not possible to look for debt contract prices to equilibrate the demand and supply for credit. Instead given a set of prices for capital, debt-contract prices will be uniquely pinned down. Proposition 2 showed that the price of debt contracts were determined by the future capital and state prices. The latter, in general, depends on the price of debt contracts and capital, given rise to a fix point problem. However, in the model presented in this paper that considers a finite time horizon, this will not be the case and debt-contract prices can be recovered by backward induction. In fact, in the final states $s_2 = UU, UD, DU, DD$ net-worth will be consumed so its marginal value is simply 1. Thus, in state $s_1 = U, D$, state prices ψ_{s_2} are given by the discount rate

[9] It is also implied by the pricing equations for capital and debt-contract $\hat{\phi}_s$, equations (17) and (16), respectively.

[10] More precisely, see Appendix A, it is a correspondence taking values, in state s, in $[0, (F(\hat{\theta}_s)\bar{n}_s - C_s)/q_{\hat{\phi}_s,s}]$.

and transition probabilities $\beta\mathbb{P}(s_2|s_1)$, and debt contract prices in state $s_1 = U, D$ are given by these state prices and the exogenous capital prices in final states $p_{UU},\ p_{UD},\ p_{DU},\ p_{DD}$.

The price of debt contracts in state 0, is given by the capital and state prices in period 1. To calculate state prices recall that form Proposition 2, state prices depend on the expected future marginal value of net-worth, and that from Lemma 1 these marginal values were characterized recursively. Using these results the following recursive characterization for state prices is obtained

$$\psi_{s'} = \left\{1 + p_{s'}\Psi'^{-1}(p_{s'}) - \Psi\left(\Psi'^{-1}(p_{s'})\right)\right\} \times \left[F(\hat{\theta}_{s'}) + \left(1 - F(\hat{\theta}_{s'})\right)\frac{\hat{\theta}_{s'} + \bar{\theta}}{2(p_{s'} - q_{\hat{\phi}_{s'},s'})}(\psi_{s'U} + \psi_{s'D})\right] \quad (20)$$

That is the price of net-worth in state s' depends on two terms. The first term–in curly brackets, represents the benefit from producing capital in state s', which will increase the value of net-worth obtained in that state. The second term–in square brackets, represents the expected discounted return that will be obtained in state s'. With probability $F(\hat{\theta}_s)$ the entrepreneur will be a lender, who obtain an expected discounted return equal to the marginal utility of consumption, equal to 1. With probability, $1 - F(\hat{\theta}_s)$, the entrepreneur will be capitalist, who earn a return equal to their productivity over the required capital downpayment, $p_{s'} - q_{\hat{\phi}_{s'},s'}$. The expected productivity of capitalists is given by $(\hat{\theta}_s + \bar{\theta})/2$ and capitalist will value this return according to their state prices in the successors of state s'. Using these expressions, given a set of prices for capital, $\{p_0, p_U, p_D\}$, debt-contracts $\hat{\phi}_s$ and state prices are pinned down. That is, expressions are obtained for $\{q_{\hat{\phi}_0,0}, q_{\hat{\phi}_U,U}, q_{\hat{\phi}_D,D}\}$ and $\{\psi_U, \psi_D, \psi_{UU}, \psi_{UD}, \psi_{DU}, \psi_{DD}\}$–where state prices corresponds to the price at the predecessor state.

To solve for equilibrium numerically I consider the values for the model parameters listed in Table 1. The baseline calibration considers a discount factor $\beta = 0.95$ and initial endowments of both capital and consumption of 1. The following values for the price of capital in the final states are considered: $p_{UU} = 1.05$, $p_{UD} = 0.9$, $p_{DU} = 0.8$, $p_{DD} = 0.2$. Note that the price of capital is higher following state U, relative to state D, and that these prices are more volatile following state D as well. A higher volatility following state D is expected to generate a lower leverage in that state (Fostel and Geanakoplos, 2011). The distribution of productivities is assumed to be uniform with support in $[0.2, 0.6]$. Depreciation is $\delta = 0.1$. The values for the coefficients of the investment cost function, $\Psi(\cdot)$, are set to zero so there will be no investment in equilibrium. Finally the conditional transition probabilities are set to $1/2$. Using the equilibrium characterization, the previous expressions, and the model parameters, I calculate the excess demand for capital in states $0, U, D$ as a function of prices $\{p_0, p_U, p_D\}$. I use a Newton method with numerical derivatives to solve for the zero of the system.

Figure 3 presents the equilibrium for the baseline parameter values listed in Table 1. As expected the price of capital is higher in state U relative to state D, $p_U = 1.3$ and $p_D = 0.9$. This reflects both the lower future values of capital and the lower LTV in state D: $LTV_U = 64\%$ and $LTV_D = 48\%$. The lower leverage is also to be expected given the higher volatility of the price of capital following state D. The price of capital in state 0 is $p_0 = 1.47$. Note that leverage is highest in state 0, despite capital price volatility being between the values in state U and D. This reflects that equilibrium interest rates are lower in state 0 relative to states U and D. In fact, the interest rate in state 0 is $r_0 = 2.1\%$, whereas the interest rate in both states U and D is $r_U = r_D = 5.3\%$

To understand the results of the numerical calibration it is useful to compare the equilibrium with the benchmark case of no financial frictions of perfect financial markets. In this case, the most productive entrepreneur will buy all the existing capital in every state and the price of capital will reflect his productivity. Every other entrepreneur will lend to him earning an expected equivalent interest rate of $1/\beta - 1 = 5.3\%$. All entrepreneurs will be indifferent between consuming in the current or future states and perceive an indirect utility equal to the value of their initial endowment, which is highest as the price of capital is highest. With no frictions the equilibrium is Pareto efficient.

In contrast, in the equilibrium of the economy studied in this paper the interest rate in state 0, is lower than 5.3%. But at the same time equals 5.3% in states U and D. What drives the wedge between the equilibrium interest rate in state 0 and $1/\beta - 1$? The reason is that lenders has the chance to be productive in the future and earn a gross return above $1/\beta$, so they are willing to lend at a lower interest rate in state 0. To see this through the equilibrium conditions consider the identity that defines the risk-free interest rate in terms of the state prices

$$1 + r_s = \frac{1}{\psi_{sU} + \psi_{sD}}$$

Since in the final period net-worth is consumed, the state prices in the terminal states is simply $\beta\mathbb{P}(s_2|s_1)$. Thus, the sum of state prices is just β and the interest rate is $r_U = r_D = 1/\beta - 1 = 5.3\%$. Instead, in state 0 the state prices reflect the investment opportunities that are expected in period 1. Recall the recursive characterization for state prices given in equation (20). In the baseline calibration there is no investment, so the state s' price is the discounted weighted average of the expected gross return of lenders and capitalists

$$\beta\left[F(\hat{\theta}_{s_1})\frac{1}{\beta} + \left(1 - F(\hat{\theta}_{s_1})\right)\frac{\hat{\theta}_{s_1} + \bar{\theta}}{2(p_{s_1} - q_{\hat{\phi}_{s_1},s_1})}\right]$$

with the expected gross return of capitalists being strictly greater than $1/\beta$. This can be seen from equation (19) that imply that the marginal entrepreneur earns a return equal to $1/\beta$, indeed

$$\frac{\hat{\theta}_{s_1}}{p_{s_1} - q_{\hat{\phi}_{s_1},s_1}} = \frac{1}{\psi_{s_1U} + \psi_{s_1D}} = \frac{1}{\beta}$$

Therefore, lenders are willing to sacrifice returns in state 0 because they expect to become productive with probability, $1 - F(\hat{\theta}_{s_1})$, and earn a return above $1/\beta$ in period 1. It follows that in equilibrium the interest rate will be smaller than 5.3% and in fact $r_0 = 2.1\%$.

In the next section I investigate what this relationship between future investment opportunities and the current interest rate imply for the inter-temporal relationship between leverage and the interest rate.

6 Comparative Statics

First, I conduct a comparative statics exercise on entrepreneurs' discount factor β. For this I solve for the equilibrium of the baseline economy, but where β takes on different values. I consider values for β in $[0.87, 0.96]$, which includes the baseline value of 0.95. Figure 4 presents the equilibrium values for the different values of β for the price of capital, leverage, interest rates, state prices and the identity of the marginal entrepreneur. As expected as entrepreneurs become more patient, i.e. β increases, interest rates drop (Panels 4c and 4d). Lower interest rates increase capital valuations and state prices (Panels 4a and 4e). Moreover, lower interest rates incentivize the use of leverage and leverage increases in all states, although the increase is very modest in period 1 (Panel 4b). A higher leverage translates into a lower downpayment for capital and through the indifference condition for the marginal entrepreneur, this will imply that the marginal entrepreneur is less productive (Panel 4f).

Second, I carry a comparative statics exercise on the lowest-exogenous-future value of capital in period 2, p_{DD}. For this I solve the model of the baseline economy, considering values for p_{DD} in $[0.1, 0.28]$. Figure 5 presents the equilibrium values for the different values of p_{DD}. As expected as the future values of capital drop in state D, the price of capital in state D drops as well (Panel 5a). In line with the results of Fostel and Geanakoplos (2010) as p_{DD} declines and price volatility increases in state D, leverage drops in state D (Panel 5b). This feedback into the price drop of capital in state D. Consequently, the price of capital also drops in state 0. On the other hand, in state D debt contract $\hat{\phi}_D$ promises the same amount $(1 - \delta)p_{DU}$, whereas the price of this contract declines, which maps into a higher implicit interest rate for risky debt in state D (Panel 5d).

More interestingly, as the lowest price of capital in period 2 declines the risk-free rate in state 0 declines, as well. To understand this effect recall that by definition the risk-free rate equals $1/(\psi_U + \psi_D) - 1$. That is, it is inversely proportional to the sum of future state prices that, as discussed above reflects the expected gross returns that entrepreneurs can obtain in those states. From equation (20) the state prices in state 0, in this case without investment, are given by

$$\psi_{s_1} = \beta\left[F(\hat{\theta}_{s_1})\frac{1}{\beta} + \left(1 - F(\hat{\theta}_{s_1})\right)\frac{\hat{\theta}_{s_1} + \bar{\theta}}{2(p_{s_1} - q_{\hat{\phi}_{s_1},s_1})}\right]$$

with the discounted expected return of capitalists being greater than 1, i.e., $\frac{(\hat{\theta}_{s_1}+\bar{\theta})}{2(p_{s_1}-q_{\hat{\phi}_{s_1},s_1})} > \frac{1}{\beta}$. Since, as p_{DD} becomes smaller, total lending decreases, then more entrepreneurs will be needed to purchase all the capital for sale, thus the marginal entrepreneur $\hat{\theta}_D$ will be a less productive entrepreneur. This imply that the probability of becoming a capitalist in state D increases. Moreover, the discounted expected return will increase, as well. This increase is due to the fact that the downpayment for capital will be smaller as the price of capital drops both due to the drop in its future value and the drop in the productivity of the marginal entrepreneur. This effect dominates the reduction in the expected productivity of capitalits. Formally, note that the discounted expected return can be expressed as

$$\frac{\beta(\hat{\theta}_D + \bar{\theta})}{2(p_D - q_{\hat{\phi}_D,D})} = \frac{1}{2}\left[1 + \frac{\bar{\theta}}{\hat{\theta}_D}\right]$$

that is decreasing in $\hat{\theta}_D$, and were I used that $p_D - q_{\hat{\phi}_D,D} = \beta\hat{\theta}_D$. Therefore, as p_{DD} declines the marginal value of net-worth in state D increases reflecting the higher expected returns that entrepreneurs expect, this leads to a lower risk-free rate in state 0, as entrepreneurs are willing to lend at lower interest rates given that they expect to make higher returns from the proceeds of risk-free loans. On the other hand, as debt contract $\hat{\phi}_0$ delivers less in state D its price will drops, increasing the risk premium in state 0 (Panels 5c and 5d).

7 Concluding Remarks

In this paper I study a dynamic model where both leverage and interest rates are determined in equilibrium. The model considers risk neutral agents, or equivalently agents with an infinite elasticity of inter-temporal substitution. In this setting without financial frictions the risk-free rate is simply given by the discount factor of agents. However, I show that the introduction of financial frictions creates a wedge between the risk-free rate and the rate of preferences in equilibrium. I consider that borrowing is limited by a collateral constraint, so arbitrage is limited and some agents enjoy risk-free returns that are higher than the equilibrium risk-free rate. In a dynamic setting the possibility of enjoying these excess returns in the future makes agents willing to lend today at lower risk-free rates relative to their rate of preferences. Excess returns are expected to be larger when leverage is expected to be lower, so a more severe expected collapse in leverage increases the wedge between the rate of preferences and the equilibrium risk-free rate, i.e. reduces the risk-free rate. At the same time, the model predicts that this will increase risk premiums on debts that default when leverage is expected to be low.

It is left to investigate how capital investment affect the relationship between future leverage and risk-free rates. A very interesting avenue for future research is to investigate the relative effects of policies designed to manipulate leverage or the interest rate.

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Appendix

A Aggregate Demand and Supply for Capital and Debt

The total land supply, from the law of motion of aggregate capital, is given by

$$K_s(p_s) = (1-\delta)K_{s^*} + \int_{\Theta} \Psi'^{-1}(p_s) n_s(\theta^*)\, dF_s(\theta) = (1-\delta)K_{s^*} + \Psi'^{-1}(p_s)\bar{n}_s$$

Contract $\hat{\phi}_s$ is supplied by capitalists. As argued in Proposition 3 these agents would like to borrow as much as possible. If $q_{\hat{\phi}_s,s} \geq \psi_{sU} p_{sU} + \psi_{sD} p_{sD}$, i.e., agents can borrow at least the future value of land given by the $\psi_{s'}$-state prices, then these agents will pledge all their capital as collateral. Capitalists collectively hold $D_k(p_s, q_{\hat{\phi}_s,s})$. On the contrary, if $q_{\hat{\phi}_s,s}$ drops below $\psi_{sU} p_{sU} + \psi_{sD} p_{sD}$ these traders would find more attractive to issue contracts with a smaller promise, which sell at the same lower price and oblige to a smaller payment in the future. Thus, the supply of $\hat{\phi}_s$ is totally inelastic and given by,

$$S_{p_{sU},s}(q_{\hat{\phi}_s,s}) = \begin{cases} 0 & \text{if } q_{\hat{\phi}_s,s} < \psi_{sU} p_{sU} + \psi_{sD} p_{sD} \\ D_k(p_s, q_{\hat{\phi}_s,s}) & \text{if } q_{\hat{\phi}_s,s} \geq \psi_{sU} p_{sU} + \psi_{sD} p_{sD} \end{cases}$$

On the other hand, the demand for contract $\hat{\phi}_s$ will come from lenders, i.e., entrepreneurs who invest in financial contracts, $\theta_s < \hat{\theta}_s$. These agents can also invest in other financial contracts that offer a discounted expected return equal to the risk-free rate. Thus, if $q_{\hat{\phi}_s,s} > \psi_{sU} p_{sU} + \psi_{sD} p_{sD}$ the demand will be zero; if $q_{\hat{\phi}_s,s} = \psi_{sU} p_{sU} + \psi_{sD} p_{sD}$ financiers will be indifferent between buying this or any other contract offering a discounted expected return equal to the risk-free rate; and if $q_{\hat{\phi}_s,s} < \psi_{sU} p_{sU} + \psi_{sD} p_{sD}$ they will strictly prefer this contract to any other. Thus, the demand for the $\hat{\phi}_s$-contract can be described by the following correspondence

$$D_{\hat{\phi}_s,s}(q_{\hat{\phi}_s,s}) \in \begin{cases} \{MLF_s/q_{\hat{\phi}_s,s}\} & \text{if } q_{\hat{\phi}_s,s} < \psi_{sU} p_{sU} + \psi_{sD} p_{sD} \\ [0, MLF_s/q_{\hat{\phi}_s,s}] & \text{if } q_{\hat{\phi}_s,s} \leq \psi_{sU} p_{sU} + \psi_{sD} p_{sD} \\ \{0\} & \text{if } q_{\hat{\phi}_s,s} > \psi_{sU} p_{sU} + \psi_{sD} p_{sD} \end{cases}$$

where MLF_s denotes the maximum loanable funds in state s and is given by the aggregate savings of lenders, who collectively also are buying all the consumption in state s, then

$$MLF_s = \int_{\underline{\theta}}^{\hat{\theta}_s} n_s(\theta_{s^*}) - c_s(\theta_s)\, dF_s(\theta) = F(\hat{\theta}_s)\bar{n}_s - C_s$$

with C_s equal to the aggregate consumption in state s, that in equilibrium will be equal to the aggregate production in the predecessor state, Y_{s^*}, or the aggregate consumption endowment in state 0.

Tables and Figures

Table 1: Baseline Calibration Parameter Values

Parameter	Description	Value
β	discount facor	0.95
e_k	initial endowment of capital	1
e_c	initial endowment of consumption	1
$[\underline{\theta}, \bar{\theta}]$	support of productivity distribution	$[0.2, 0.6]$
δ	capital depreciation	0.1
α_1	investment cost parameter	0
$1/\alpha_2$	investment cost parameter	0
π	transition probability	0.5
p_{UU}	capital value in state UU	1.05
p_{UD}	capital value in state UD	0.90
p_{DU}	capital value in state DU	0.80
p_{DD}	capital value in state DD	0.20

Notes: See the text for a more detailed description of the parameters.

Figure 1: Tree of Events.

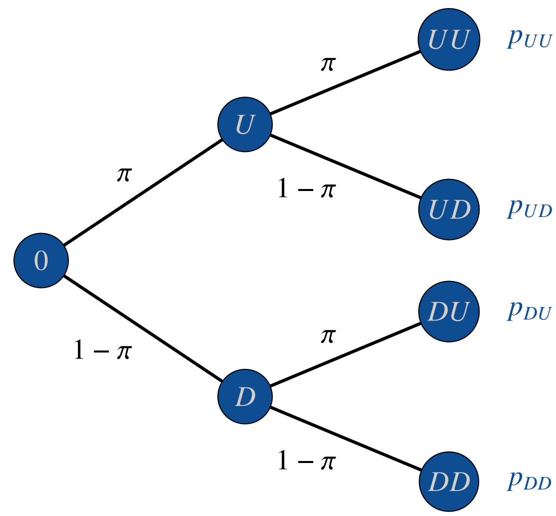


Figure 2: Entrepreneurs' Optimal Behavior in State s.

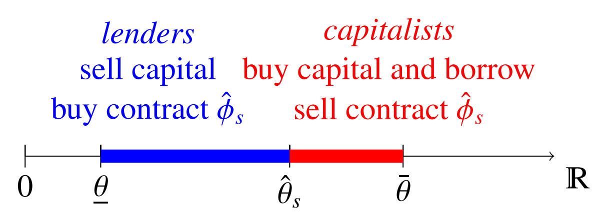
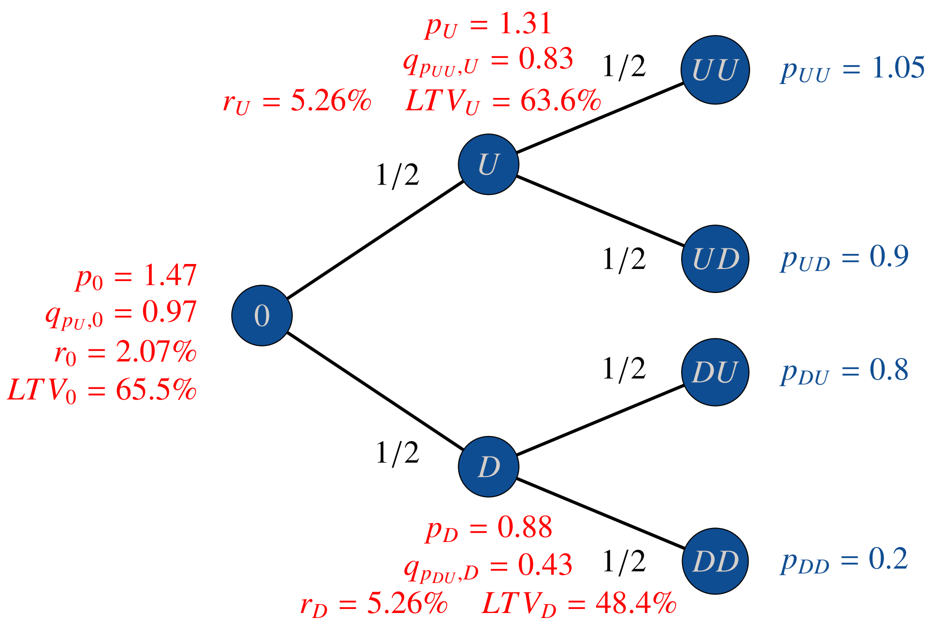
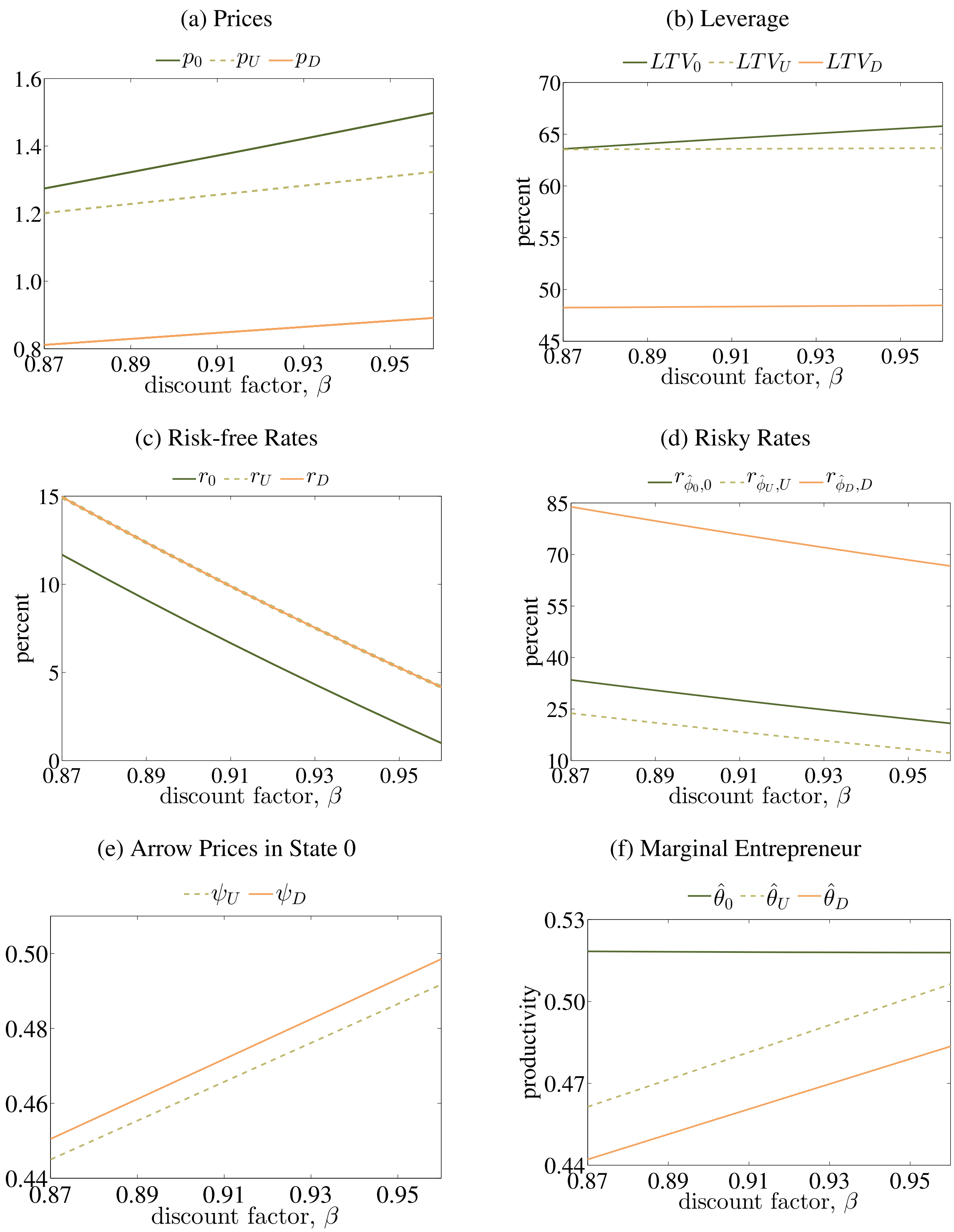


Figure 3: Equilibrium of the Baseline Calibration Model



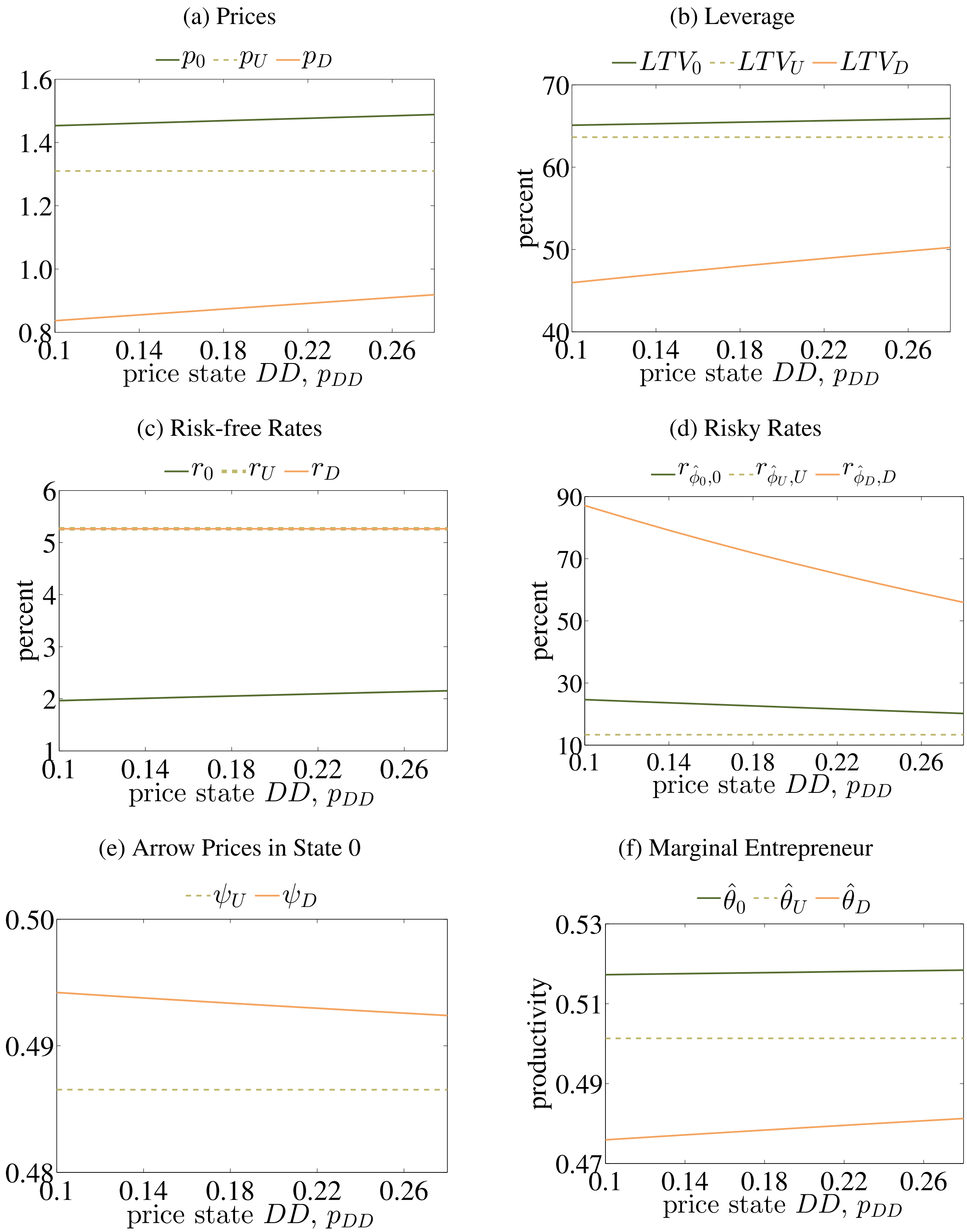
Notes: Equilibrium corresponds to the baseline model calibration, see Table 1 for the values of the parameters used.

Figure 4: Comparative Statics on Entrepreneurs' Discount Factor β



Notes: Equilibrium values correspond to the baseline model calibration for all the parameteres except the discount factor β that takes values from 0.90 to 0.99. See Table 1 for the values of the parameters used.

Figure 5: Comparative Statics on Capital Price in State DD, p_{DD}



Notes: Equilibrium values correspond to the baseline model calibration for all the parameteres except the price of capital in state DD, p_{DD}, which takes values from 0.10 to 0.28. See Table 1 for the values of the parameters used.