On the Programmability and Uniformity of Digital Currencies

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Abstract

Central bankers often suggest that programmable money may not be desirable, as it could compromise the uniformity, or singleness, of money. We explore this concern in a simple model that endogenizes the creation of programmable money and the liquidity values of differently programmed currencies (i.e., the degree of singleness). Programmable money is privately valuable due to commitment frictions. However, its creation can be socially costly in the presence of informational frictions in recognizing different types of money. We demonstrate that preserving the singleness of money is not necessarily socially beneficial. Moreover, the prohibition of programmable money reduces social welfare when informational frictions are mild, and enhances welfare when commitment frictions are negligible.

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*We thank many friends and colleagues for many useful conversations. The views expressed here are those of the authors and do not necessarily reflect the position of the Bank of Canada.

1 Introduction

Many researchers and practitioners believe that a truly special feature of digital money, compared to traditional money, is its programmability. This characteristic allows for the embedding of software programs into digital money to automatically enforce pre-specified rules when certain conditions are met, eliminating the need for human intervention or intermediaries. Today, programmable money are often implemented by blockchain-based smart contracts that execute state-dependent transactions of monetary values recorded on a distributed ledger.[1] For example, a smart contract can execute or block a payment transfer according to pre-defined conditions (e.g., delivery or expiry date). Kahn and von Oordt (2024) have even argued that it may be optimal to program an expiry date after which the digital currency cannot be transferred and so looses its value. Proponents argue that programmable money has the potential to revolutionize financial transactions by overcoming commitment problems, automating transactions, and improving efficiency. Some central banks also incorporate programmability into the design of their central bank digital currencies.[2]

However, money balances that are programmed differently may possess distinct market values reflecting their associated transferability, maturity, contractibility and risk. This variation could lead to a deviation from the concept of the uniformity or "singleness" of the money stock, where each unit of currency should maintain the same value and purchasing power as any other unit of the same denomination. The nature and the importance of the singleness of money is discussed in a recent BIS report by Garratt and Shin (2023):

> *"Singleness ensures that monetary exchange is not subject to fluctuating exchange rates between different forms of money, whether privately issued (e.g., deposits) or publicly issued (e.g., cash). With singleness of money, an unambiguous unit of account supports all economic transactions in society."*

[1] A programmable money based on smart contracts define the rules of the monetary system in the form of computer scripts. These rules can be programmed in terms of state variables such as time, location, payment size, origin, and destination.

[2] For example, the Banco Central do Brasil published guidelines for the implementation of Digital Real which emphasize "the development of innovative models with the incorporation of technologies, such as smart contracts and programmable money" (https://www.bcb.gov.br/detalhenoticia/667/noticia.). The e-CNY project also allows programmability that "enables self-executing payments according to predefined conditions or terms agreed between two sides, so as to facilitate business model innovation." (http://www.pbc.gov.cn/en/3688110/3688172/4157443/4293696/2021071614584691871.pdf). See Chiu, Kahn and Koeppl (2022) and Lee et al. (2024) for how tokenization and smart contracts in general can overcome commitment problems.

"However, creditworthiness alone is typically insufficient to maintain singleness, as singleness relies on the universally shared confidence in the value of money among users. Even a minor seed of doubt, whether justified or not, can ripple through monetary exchanges and potentially undermine money's role as a medium of exchange. In this context, 'moneyness' extends beyond merely the absence of credit risk."

"To be clear, singleness of money does not preclude varying credit risks across intermediaries (...) Singleness attributes to the payment, rather than to private liabilities as a store of value." Garratt and Shin (2023)

This is an important consideration for the design of digital money. Indeed, some central bankers have expressed concerns that the introduction of programmable money could threaten the singleness of money.[3]

"If programmability becomes a highly desired feature of money, and its supply is freely developed, it could jeopardize the singleness of money. This might lead to the use of multiple units of account in parallel within a country, or to a more fragile parity between different representations of the same unit of account." Speech by Ida Wolden Bache (Governor of Norges Bank)

Given the innovative potential of programmability and its profound implications for maintaining the singleness of money, fundamental research is required to guide policy decisions. However, there are currently no formal economic theories to answer questions on the interactions between the programmability and the singleness of money such as:

- Why is programmable money beneficial?

[3] Relatedly, Carolyn A. Wilkins, an external member of Bank of England Financial Policy Committee, argued in a speech: "programming could also undermine the uniformity of money which is required to provide a safe base to the financial system. One could easily imagine that a CBDC that had programmed restrictions would become a less preferred means of payment than other forms of money. It makes sense, therefore, that a digital pound would not include any government or central bank-initiated programmable features, although users could set up their own programmable payments if they wanted." (https://wwwtest.bankofengland.co.uk/speech/2023/may/carolyn-wilkins-keynote-speech-at-omfif). Furthermore, the 2023 Consultation Paper by the Bank of England and the HM Treasury: "We do not propose to develop a digital pound that enables government or central bank-initiated programmable money. As discussed in Part B, payments programmability could provide enhanced functionality for users to set rules on their payments. While it may be possible to program the digital pound so that it could only work in certain ways, this is not relevant to HM Treasury and the Bank's policy objectives for the digital pound. Further, this functionality could damage the uniformity of the CBDC and cause user distrust." (https://www.bankofengland.co.uk/-/media/boe/files/paper/2023/the-digital-pound-consultation-working-paper.pdf)

- Why is the singleness of money important?
- Under which conditions does programmability conflict with singleness?
- Should programmability be restricted to preserve the singleness of money?

This paper addresses these questions by developing a microfounded model of programmable money. The environment below features agents with absence of double coincidence of wants so that a means of payment is needed. Banks provide this means of payment which can be programmed to be more or less liquid. Some buyers (L) are more likely to consume early and it is inefficient that they consume late but they cannot commit to refrain from consuming late. It is therefore efficient that those buyers use programmed money. Some other buyers (H) are less likely to consume early than late, however, it does not matter for efficiency when they consume. Those buyers do not care in general for programmed money. Sellers also have some preference shocks and some (early sellers) would like to consume rather earlier than later. Those sellers who want to consume earlier will have no lust for programmed money since it cannot be spent, the other sellers (late sellers) will be indifferent. Optimally, L buyers should meet late sellers, and H sellers should trade with early sellers. But sellers do not know their types yet when they trade with buyers. In addition, we will also introduce private information as another layer of inefficiency: the buyer's type is private information and some sellers cannot recognize programmed money from other types of money. Our model endogenizes the creation of programmable money, and the liquidity values of differently programmed monies (i.e., the degree of singleness). We use this framework to evaluate the societal benefits of both programmability and singleness. We investigate the impact of informational (represented by parameter π) and commitment frictions (represented by σ_L) on the equilibrium creation of programmable monies and the degree of singleness, clarifying under what circumstances the prohibition of programmability could enhance welfare.

Our study also suggests that singleness is not a necessary condition for optimality. Indeed, an intervention that forces all monies to be traded at par, achieving perfect singleness, can be damaging as the Gresham's law can apply with "bad" money driving out "good" ones.

The idea that the stock of money should remain uniform is not new but advances in new technology makes it easier to program money, which brings some benefits. In addition, unlike traditional payment services e.g., PayPal, blockchain-based cryptocurrencies bundle programmability with the digital representation of money (Lee, 2021), implying that differently programmed cryptocurrencies cannot have fungible digital representations...

2 Environment

We consider a finite-horizon model with four consecutive periods.[4] In the first and the fourth periods, agents trade in a centralized market (CM). In the second and third periods, agents are subject to bilateral matching in a decentralized market. Hence, the four consecutive periods are denoted CM1, DM1, DM2 and CM2.There are a large number of short-lived agents: buyers, sellers and bankers. Buyers and bankers enter the economy in CM1 and buyers exit at the end of the following DM2, while bankers exit at the end of the following CM. Sellers enter the economy in DM1 and DM2 and exit at the end of the following CM. The discount factor between CM1 and DM1 is β.[5]

Buyers

A buyer in CM1 can be of type $i = L$ or type $i = H$ with $Pr(i = L) = f_L$ and $Pr(i = H) = f_H = 1 - f_L$. A type $i = H, L$ buyer is subject to a shock that determines whether the buyer values consumption in DM1 or DM2. Let $q_1$be the consumption in DM1 and q_2 the consumption in DM2. Then the preferences of buyer i is

$$U_i(q_1, q_2) = \eta_i u(q_1) + (1 - \eta_i) u_i(q_2)$$

where $\eta_i \in \{0, 1\}$ is a stochastic variable that takes value 1 with probability σ_i and 0 otherwise, and where

$$u'(q) > 0, u''(q) < 0, u'(0) = \infty$$
$$u_L(q) = \varepsilon q$$
$$u_H(q) = u(q)$$

Hence, type L buyers derive utility $u(q_1)$ from consumption in DM1 with probability σ_L and derive utility εq_2 with $\varepsilon < 1$ from consumption in DM2 with probability $1 - \sigma_L$; Type H buyers derive utility $u(q_1)$ from consumption in DM1 with probability σ_H and derive utility $u(q_2)$ from consumption in DM2 with probability $1 - \sigma_H$. Also, assume $\sigma_L > \sigma_H$. Therefore, L buyers are more likely to consume early than H buyers, but in case they consume late, their marginal utility is (arbitrarily) low.

Sellers

There are two types of sellers. Some sellers enter the economy in DM1 and others enter in DM2. Sellers in DM1 produce q_1 with a linear cost function. They are subject to a consumption shock. With

[4] It is straightforward to present our model in a standard, infinite-horizon monetary framework such as Lagos and Wright (2005) by repeating the finite economy in an overlapping-generations fashion.

[5] The timing and the size of discounting do not matter but introducing it facilitates the economic interpretation of the model as it matches the feature in a standard monetary model such as Lagos and Wright (2005).

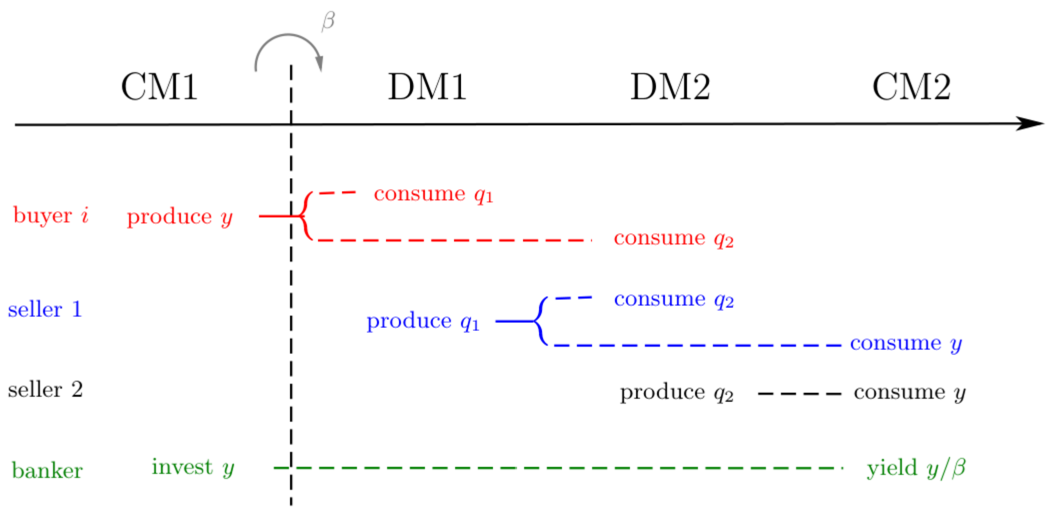

Figure 1: Model Setup

probability α_e, the seller is an early (e) seller who wants to consume in DM2 with a linear utility. With probability $1-\alpha_e$, the seller is a late (ℓ) type who wants to consume in the CM with a linear utility function. The type is not known at the time of DM1 production but it is randomly decided *after* the DM1. Sellers in DM2 produce q_2 with a linear cost function and want to consume in CM with a linear utility function. The terms-of-trade in DM1 and DM2 are determined by TIOLI offers from the buyers.

Bankers

Bankers create and sell tokens in one CM for ϕ numeraire goods and keep reserve of the numeraire good to redeem them in the following CM. We assume that reserves generate a return rate $1/\beta$ (this implies that CIA does not distort consumption). The tokens can be programmed so that it is transferable in DM2 with probability $\mathfrak{p} \in \{0,1\}$. The price at which the bank sells the token is thus a function of $\mathfrak{p}$: $\phi_{\mathfrak{p}}$. The token can always be transferred in DM1. However, in DM2, the token can either be programmed ($\mathfrak{p}=1$) so that it is not transferableor it is not programmed ($\mathfrak{p}=0$) and it is always transferable.. The banker can commit to the promise that the bearer of a token can redeem it for one unit of numeraire goods in the next CM. The banker is subject to a balance-budget constraint that the numeraire good reserve needs to be sufficient for redemption. Buyers of type i can purchase any portfolio (m_{i0}, m_{i1}) constituted of $\mathfrak{p}=0$ tokens and $\mathfrak{p}=1$ tokens respectively.

First best benchmark

It is straightforward to compute the first best allocation in this economy. In DM1, all buyers should consume q^* solving $u'(q^*) = 1$. In DM2, H-buyers should consume q^*, while L-buyers should not consume since their marginal utility is always lower than the marginal cost of production. All other consumption levels are indeterminate.

Finally, we define the meaning of singleness and programmability in our simple setup. We use $\mathcal{M}_i$ to denote the set of tokens that buyer i is holding in equilibrium. Then $\mathcal{M}_i \subset \{\{0,1\},\{0\},\{1\}\}$ for $i = H, L$, where for example $\mathcal{M}_H = \{0,1\}$ and $\mathcal{M}_L = \{1\}$ means that H buyers hold a portfolio of both $\mathfrak{p} = 1$ and $\mathfrak{p} = 0$ tokens, while L-buyers only hold $\mathfrak{p} = 1$ tokens.

Definition 1. The degree of singleness, $\mathcal{S}$, measures the fraction of meetings where all balances created in equilibrium, $\mathcal{M}_L \cup \mathcal{M}_H$, are valued the same. The degree of programmability, $\mathcal{P}$, measures the fraction of balances created in equilibrium with $\mathfrak{p} = 1$:

$$\mathcal{P} = \frac{f_L m_{L1} + f_H m_{H1}}{\sum_{\mathfrak{p}=0,1} f_L m_{L\mathfrak{p}} + f_H m_{H\mathfrak{p}}}.$$

3 Equilibrium Characterization with Perfect Recognizability

With probability α_e, the seller in DM1 wants to consume in DM2 and with probability $1-\alpha_e$, the seller wants to consume in the CM2. The type is known only ex-post. So a portfolio $\mathbf{m}_i = (m_{i0}, m_{i1})$ can induce the seller to sell $q_{i1} = m_{i1}(1-\alpha_e) + m_{i0}$. We now derive the transaction values of tokens in each market. In DM2, each token $\mathfrak{p}$ can buy $1-\mathfrak{p}$ units of q_2. In particular, since $\mathfrak{p} = 1$ tokens are not transferable, they cannot buy anything in DM2. In DM1, each token $\mathfrak{p}$ transferred to a seller can buy $1 - \alpha_e + \alpha_e(1-\mathfrak{p})$ unit of q_1.

The demand for $\mathfrak{p}$-tokens To derive the demand for tokens, we first determine the marginal value of a token $\mathfrak{p}$ to a type L buyer in the DM,

$$\frac{\partial v_L'(\mathbf{m}_i)}{\partial m_{i\mathfrak{p}}} = \sigma_L u'(q_{1L})[1-\alpha_e+\alpha_e(1-\mathfrak{p})] + (1-\sigma_L)\varepsilon(1-\mathfrak{p}),$$

where the buyer buys $1-\alpha_e+\alpha_e(1-\mathfrak{p})$ units from a seller in DM1 w.p. σ_L, and buys $(1-\mathfrak{p})$ units from a seller in DM2 w.p. $1-\sigma_L$ to derive marginal utility ε. In the CM, given the price of token $\mathfrak{p}$, $\phi_{\mathfrak{p}}$, a type L buyer's FOC wrt the quantity of $m_{i\mathfrak{p}}$ is

$$-\phi_{\mathfrak{p}} + \frac{\partial v_L'(\mathbf{m}_i)}{\partial m_{i\mathfrak{p}}} \leq 0.$$

The marginal value of a token $\mathfrak{p}$ to a type H buyer in the DM is

$$\frac{\partial v_H'(\mathbf{m}_i)}{\partial m_{i\mathfrak{p}}} = \sigma_H u'(q_{1H})[1-\alpha_e+\alpha_e(1-\mathfrak{p})] + (1-\sigma_H)u'(q_{2H})(1-\mathfrak{p}),$$

where the buyer buys $1-\alpha_e+\alpha_e(1-\mathfrak{p})$ unit from a seller in DM1 w.p. σ_1, and buys $(1-\mathfrak{p})$ units from a seller in DM2 w.p. $1-\sigma_L$. In the CM, given the price of token $\mathfrak{p}$, $\phi_{\mathfrak{p}}$, a type H buyer's FOC wrt the quantity of $m_{i\mathfrak{p}}$ is

$$-\phi_{\mathfrak{p}} + \frac{\partial v_H'(\mathbf{m}_i)}{\partial m_{i\mathfrak{p}}} \leq 0.$$

The bank supply of $\mathfrak{p}$-tokens Being risk neutral, the bank will supply any amount of $\mathfrak{p}$-tokens to buyers as long as it makes a non-negative profit. We denote by $\phi_{\mathfrak{p}}$ the real revenue of selling a $\mathfrak{p}$-token. For each token, the bank invests $\phi_{\mathfrak{p}}$, which generates $\phi_{\mathfrak{p}}/\beta$ in the last CM. With this revenue, the bank has to fulfill its promise to redeem token holders. Therefore the bank's problem pins down the price of each $\mathfrak{p}$-token, as a function of the number of tokens that will be redeemed.

If a $\mathfrak{p}$-token is held solely by type i buyers, then the expected redemption is

$$\sigma_i[1-\alpha_e+\alpha_e(1-\mathfrak{p})] + (1-\sigma_i)(1-\mathfrak{p}),$$

and the zero-profit condition of the banker gives

$$\phi_{\mathfrak{p}} = \beta(\sigma_i(1-\alpha_e+\alpha_e(1-\mathfrak{p})) + (1-\sigma_i)(1-\mathfrak{p})). \tag{1}$$

Otherwise, if a $\mathfrak{p}$-token is held by both types, then the zero-profit condition of the banker is given by

$$\phi_{\mathfrak{p}} = \beta\frac{1}{f_L m_{L\mathfrak{p}} + f_H m_{H\mathfrak{p}}} \sum_{i=L,H} \rho_i m_{i\mathfrak{p}}[\sigma_i(1-\alpha_e+\alpha_e(1-\mathfrak{p})) + (1-\sigma_i)(1-\mathfrak{p})]. \tag{2}$$

Proposition 1. *With $\varepsilon < 1$ and $\sigma_L > \sigma_H$, the unique equilibrium with perfect recognizability is such that type L buyers only hold $\mathfrak{p} = 1$-tokens and type H buyers only hold $\mathfrak{p} = 0$-tokens. The equilibrium allocation does not depend on the value of α_e.*

With perfect recognizability, sellers know which token they are purchasing from buyers. Therefore, sellers are producing more for $\mathfrak{p} = 0$-tokens than for $\mathfrak{p} = 1$-tokens. This would induce L-buyers to hold $\mathfrak{p} = 0$-tokens. However, when they purchase tokens from the bank, $\mathfrak{p} = 0$ tokens are more expensive than $\mathfrak{p} = 1$-tokens because they are redeemed more often. As L-sellers derive little utility in CM2 ($\varepsilon < 1$) and given they do not consume in CM2 often ($\sigma_L > \sigma_H$), they prefer to purchase the cheaper $\mathfrak{p} = 1$-tokens, that better accommodate their (expected) consumption needs.

Prohibition of programmable money

Next, we consider the case where the regulator prohibits programmable money . Therefore there are only tokens with $\mathfrak{p} = 0$, which is bank deposit as we know it (it can be transferred at will). The equilibrium conditions for the L and H type buyers, and the banker are given by:

$$\begin{aligned} \phi_0 &= \beta\sigma_L u'(q_{1L}) + \beta(1-\sigma_L)\varepsilon, \\ \phi_0 &= \beta u'(q_{1H}), \\ \phi_0 &= \beta. \end{aligned}$$

Since $\varepsilon < 1$, this then implies that $u'(q_{1H}) = 1 < u'(q_{1L})$. Given H-buyers have a larger marginal utility of consumption in both DMs relative to L-buyers, they are willing to pay more than L-buyers for $\mathfrak{p} = 0$-tokens. Therefore L-sellers find these tokens too expensive, they do not acquire as many than H-buyers and as a result, they reduce their consumption. Therefore the equilibrium allocation without programmability cannot be optimal. We summarize the results as follows,

Proposition 2. *With perfect recognizability, the first-best allocation is supported with* $\mathcal{S} = 0$ and $\mathcal{P} = \frac{f_L}{f_L + f_H(1-\alpha_e)}$. *Prohibiting programmability makes type L buyers worse off, reducing social welfare, even though it can fully restore the singleness of money. In this sense, programmability is essential.*

Interestingly, prohibiting programmable money hurts those L-buyers who, we would think, benefit from being able to spend it in DM2 (relative to programmable tokens). The reason is a pecuniary externality of sort: Banks anticipate that all their tokens will be redeemed and they respond by increasing the price (ϕ_0). As a consequence, L buyers are worse off, since that price is too high relative to their marginal value of consumption in DM2.

4 Imperfect Recognizability

Next we consider the case where tokens can be programmed, but we now assume that a fraction π of DM1 sellers cannot observe the token characteristic $\mathfrak{p}$. Those sellers who cannot observe $\mathfrak{p}$ may want to use contractual terms to learn them. However, this is not possible, because all buyers active in DM1 (selling their token) derive the same utility from consumption. Faced with an unknown token, it is natural to assume that sellers use the population average to value that token. So a unit of (unknown) token can induce an uninformed seller to sell $q_1^{\pi} = (1-\alpha_e) + \alpha_e(1-\tilde{\mathfrak{p}})$ units of DM1 goods, which is the expected payoff of consuming a unit in the CM plus the expected payoff of consuming in DM2 given that the fraction of transferable tokens is

$$1 - \tilde{\mathfrak{p}} = 1 - \mathcal{P} = \frac{f_L m_{L0} + f_H m_{H0}}{\sum_{\mathfrak{p}=0,1} f_L m_{L\mathfrak{p}} + f_H m_{H\mathfrak{p}}}.$$

Then the marginal value of a token $\mathfrak{p}$ to a type L buyer in the DM is

$$\begin{aligned}\frac{\partial v_L(\mathbf{m}_L)}{\partial m_{L\mathfrak{p}}} &= \sigma_L u'(q_{1L})(1-\pi)\left[1-\alpha_e+\alpha_e(1-\mathfrak{p})\right] \\ &\quad +\sigma_L u'(q_{1L}^{\pi})\pi\left[1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})\right]+(1-\sigma_L)\varepsilon(1-\mathfrak{p}).\end{aligned}$$

In the CM, given the price of token $\mathfrak{p}$, $\phi_{\mathfrak{p}}$, a type L buyer's FOC wrt the quantity of $m_{L\mathfrak{p}}$ is

$$-\phi_{\mathfrak{p}} + \frac{\partial v_L(\mathbf{m}_L)}{\partial m_{L\mathfrak{p}}} \leq 0.$$

The marginal value of a token $\mathfrak{p}$ to a type H buyer in the DM is

$$\begin{aligned}\frac{\partial v_H(\mathbf{m}_H)}{\partial m_{H\mathfrak{p}}} &= \sigma_H u'(q_{1H})(1-\pi)\left[1-\alpha_e+\alpha_e(1-\mathfrak{p})\right] \\ &\quad +\sigma_H u'(q_{1H}^{\pi})\pi\left[1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})\right]+(1-\sigma_H)u'(q_{2H})(1-\mathfrak{p}).\end{aligned}$$

In the CM, given the price of token $\mathfrak{p}$, $\phi_{\mathfrak{p}}$, a type H buyer's FOC wrt the quantity of $m_{H\mathfrak{p}}$ is

$$-\phi_{\mathfrak{p}} + \frac{\partial v_H(\mathbf{m}_H)}{\partial m_{H\mathfrak{p}}} \leq 0.$$

Next we show that it is still an equilibrium that type L-buyers only hold $\mathfrak{p} = 1$-tokens and type H-buyers only hold $\mathfrak{p} = 0$-tokens.

Proposition 3. *By continuity, for π not too high, $\mathcal{M}_H = \{0\}$, $\mathcal{M}_L = \{1\}$ is an equilibrium of the economy with imperfect recognizability.*

The result in Proposition 3 is intuitive. If the adverse selection problem in DM1 is not too severe (π is small), the expected price of a token will not differ too much from the price of the same token in the equilibrium with perfect recognizability. Therefore, the incentives of H and L buyers to only hod one type of token are preserved.

Proof. Consider an equilibrium with $\mathcal{M}_{\mathrm{H}} = \{0\}$,$\mathcal{M}_{\mathrm{L}} = \{1\}$. The equilibrium conditions for the type L- and H-buyers, and the bank are now given by:

$$
\begin{aligned}
\phi_1 &= \beta\sigma_L u'(q_{1L})(1-\pi)(1-\alpha_e) + \beta\sigma_L u'(q_{1L}^{\pi})\pi\left[1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})\right], \\
\phi_0 &= \beta\sigma_H u'(q_{1H})(1-\pi) + \beta\sigma_H u'(q_{1H}^{\pi})\pi\left[1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})\right] + \beta(1-\sigma_H)u'(q_{2H}), \\
\phi_0 &= \beta, \\
\phi_1 &= \beta\sigma_L(1-\alpha_e), \\
1-\tilde{\mathfrak{p}} &= \frac{f_H m_{H0}}{f_L m_{L1} + f_H m_{H0}}.
\end{aligned}
$$

Since H-buyers only hold $\mathfrak{p}=0$-tokens that are transferable in both DMs, the consumption in DM1 when they meet an informed seller is equal to their consumption in DM2, $q_{1H}=q_{2H}\equiv q_H$. Since they hold m_{H0} units of $\mathfrak{p}=0$-tokens, $q_H=m_{H0}$. However, when they meet an uninformed seller in DM1, they can only obtain $m_{H0}\left[1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})\right]$ units of consumption. Therefore, we have

$$
q_{1H}=q_{2H}=\frac{q_{1H}^{\pi}}{1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})} > q_{1H}^{\pi}.
$$

In contrast, for L-buyers who only hold $\mathfrak{p}=1$-tokens, their consumption in DM1 is higher when they meet an uninformed seller (relative to when they meet an informed one) because that uninformed seller values the token at the average value. Therefore

$$
q_{1L}=m_{L1}(1-\alpha_e) < m_{L1}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) = q_{1L}^{\pi}.
$$

To verify that this is an equilibrium, we need to check that the first order conditions are satisfied. First, L-buyers should have no incentives to hold $\mathfrak{p}=0$-tokens, that is

$$
\phi_0 > \frac{\partial v_L(\mathbf{m}_L)}{\partial m_{L0}},
$$

or

$$
1 > \sigma_L u'(q_{1L})(1-\pi) + \sigma_L u'(q_{1L}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + (1-\sigma_L)\varepsilon. \tag{3}
$$

Note that the FOC of L-buyers given ϕ_1 implies that

$$
\begin{aligned}
&\sigma_L u'(q_{1L})(1-\pi) + \sigma_L u'(q_{1L}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\
=&\sigma_L(1-\alpha_e) + \alpha_e\sigma_L u'(q_{1L})(1-\pi).
\end{aligned}
$$

Using this result, inequality (3) becomes

$$
1 > \sigma_L(1-\alpha_e) + \alpha_e\sigma_L u'(q_{1L})(1-\pi) + (1-\sigma_L)\varepsilon. \tag{4}
$$

Also, from the low type's FOC, we know that

$$\beta\sigma_L(1-\alpha_e) > \beta\sigma_L u'(q_{1L})(1-\pi)(1-\alpha_e),$$

which implies that

$$1 > u'(q_{1L})(1-\pi).$$

Therefore we can bound the RHS of(4),

$$\begin{aligned}
&\sigma_L(1-\alpha_e) + \alpha_e\sigma_L u'(q_{1L})(1-\pi) + (1-\sigma_L)\varepsilon \\
<&\sigma_L(1-\alpha_e) + \alpha_e\sigma_L + (1-\sigma_L)\varepsilon \\
=&\sigma_L + (1-\sigma_L)\varepsilon \\
<&1.
\end{aligned}$$

This shows that (4) and therefore (3) always hold. Hence L-buyers have no incentives to hold $\mathfrak{p} = 0$-tokens. Next, we also need to show that H-buyers have no incentive to hold $\mathfrak{p} = 1$-tokens. This is the case if π is low enough so that

$$\phi_1 > \beta\sigma_H u'(q_{1H})(1-\pi)(1-\alpha_e) + \beta\sigma_H u'(q_{1H}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})).$$

□

Proposition 4. *In the equilibrium with $\mathcal{M}_H = \{0\}$ and $\mathcal{M}_L = \{1\}$ prohibiting programmability is welfare-reducing for π not too large, and is welfare-improving for σ_L not too small.*

Proof. When $\pi = 0$, the first-best allocation is supported with programmability, and not supported without it. By the continuing existence of the $\mathcal{M}_{\mathrm{H}} = \{0\}, \mathcal{M}_{\mathrm{L}} = \{1\}$ equilibrium with imperfect recognizability when π is not too large, welfare with programmability is close to the first best welfare with $\pi = 0$. In particular, prohibiting programmability would create a first order loss in informed meeting by moving away from the first allocations in those meetings, and would generate only a second order gain by shifting the allocation in uninformed meetings, because there are few uninformed meetings when π is small. We now examine the effect of σ_L in this equilibrium. When there is no programmability, the social welfare is

$$\tilde{W} = f_L[\sigma_L W_{1L} + (1-\sigma_L)W_{2L}] + f_H[\sigma_H W_{1H} + (1-\sigma_H)W_{2H}]$$

where

$$W_{1L} = u(q_{1L}) - q_{1L},$$
$$W_{2L} = q_{2L}(\varepsilon - 1),$$
$$W_{1H} = u(q_{1H}) - q_{1H},$$
$$W_{2H} = u(q_{2H}) - q_{2H}.$$

The equilibrium conditions imply

$$\phi_0 = \beta\sigma_L u'(q_{1L}) + \beta(1-\sigma_L)\varepsilon,$$
$$\phi_0 = \beta\sigma_H u'(q_{1H}) + \beta(1-\sigma_H)u'(q_{2H}),$$
$$\phi_0 = \beta.$$

Hence, we have

$$1 = u'(q_{1H}) = u'(q_{2H}) = \sigma_L u'(q_{1L}) + (1-\sigma_L)\varepsilon,$$

implying that $q_{1H} = q_H^*$ and q_{1L} is arbitrarily close to q_{1L}^* as $\sigma_L \to 1$. As a result, $\tilde{W}$ approaches its first-best level as $\sigma_L \to 1$.

When $\sigma_L = 1$, the welfare with programmability is

$$\bar{W} = f_L[(1-\pi)W_{1L} + \pi W_{1L}^{\pi}] + f_H[\sigma_H(1-\pi)W_{1H} + \sigma_H\pi W_{1H}^{\pi} + (1-\sigma_H)W_{2H}]$$

with

$$W_{1L} = u(q_{1L}) - q_{1L}$$
$$W_{1L}^{\pi} = u(q_{1L}^{\pi}) - q_{1L}^{\pi}$$
$$W_{1H} = u(q_{1H}) - q_{1H}$$
$$W_{1H}^{\pi} = u(q_{1H}^{\pi}) - (q_{1H}^{\pi})$$
$$W_{2H} = u(q_{2H}) - q_{2H}$$

Note that

$$q_{1L} = m_{L1}(1-\alpha_e)$$
$$q_{1L}^{\pi} = m_{L1}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$
$$q_{1H} = m_{H0}$$
$$q_{2H} = m_{H0}$$
$$q_{1H}^{\pi} = m_{H0}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$

implying that

$$q^{\pi}_{1L} = q_{1L}\frac{1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})}{1-\alpha_e}$$
$$q^{\pi}_{1H} = q_{1H}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) = q_{2H}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})).$$

Hence, whenever $\tilde{\mathfrak{p}} < 1$, some quantities are not at their first-best levels. Therefore, $\bar{W}$ is lower than its first-best level when $\sigma_L = 1$. By continuity, this is true for $\sigma_L$close to 1

□

5 Special Cases

Example with Log Utility

As shown in the next section, for CRRA preference in general, we only need to consider cases (i) and (iv).

Case (i) $\mathcal{M}_{\mathrm{H}} = \{0\}$,$\mathcal{M}_{\mathrm{L}} = \{1\}$

The equilibrium conditions for the L type, H type and the banker are given by:

$$\phi_1 = \beta\sigma_L\frac{(1-\pi)}{q_{1L}}(1-\alpha_e) + \beta\sigma_L\frac{\pi}{q^{\pi}_{1L}}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$
$$\phi_0 = \beta\sigma_H\frac{(1-\pi)}{q_{1H}} + \beta\sigma_H\frac{\pi}{q^{\pi}_{1H}}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta\frac{(1-\sigma_H)}{q_{2H}}$$
$$\phi_0 = \beta$$
$$\phi_1 = \beta\sigma_L(1-\alpha_e)$$
$$1-\tilde{\mathfrak{p}} = \frac{f_H m_{H0}}{f_L m_{L1} + f_H m_{H0}}$$

Equilibrium quantities and prices

$$q_{1L} = m_{L1}(1-\alpha_e)$$
$$q^{\pi}_{1L} = m_{L1}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$
$$q_{1H} = m_{H0}$$
$$q_{2H} = m_{H0}$$
$$q^{\pi}_{1H} = m_{H0}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$

$$m_{L1} = \frac{1}{1-\alpha_e}$$

$$m_{H0} = 1$$

$$1 - \tilde{\mathfrak{p}} = \frac{f_H}{\frac{f_L}{1-\alpha_e} + f_H}$$

$$\phi_0 = \beta$$

$$\phi_1 = \beta \sigma_L (1-\alpha_e)$$

Type H has no incentives to hold $\mathfrak{p} = 1$-tokens if

$$\sigma_L(1-\alpha_e) > \sigma_H(1-\pi)(1-\alpha_e) + \sigma_H \pi.$$

$$\sigma_L(1-\alpha_e) > \sigma_H(1-\alpha_e) + \sigma_H \pi (1-(1-\alpha_e)).$$

$$\frac{(\sigma_L - \sigma_H)(1-\alpha_e)}{\sigma_H \alpha_e} > \pi.$$

Case (iv) $\mathcal{M}_{\mathrm{H}} = \{0,1\}$,$\mathcal{M}_{\mathrm{L}} = \{1\}$

The equilibrium conditions for the L- and H-buyers and the bank are given by:

$$\phi_1 = \beta\sigma_L \frac{(1-\pi)}{q_{1L}}(1-\alpha_e) + \beta\sigma_L \frac{\pi}{q_{1L}^{\pi}}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$

$$\phi_1 = \beta\sigma_H \frac{(1-\pi)}{q_{1H}}(1-\alpha_e) + \beta\sigma_H \frac{\pi}{q_{1H}^{\pi}}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$

$$\phi_0 = \beta\sigma_H \frac{(1-\pi)}{q_{1H}} + \beta\sigma_H \frac{\pi}{q_{1H}^{\pi}}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta\frac{(1-\sigma_H)}{q_{2H}}$$

$$\phi_0 = \beta$$

$$\phi_1 = \beta\frac{\sigma_L f_L m_{L1} + \sigma_H f_H m_{H1}}{f_L m_{L1} + f_H m_{H1}}(1-\alpha_e)$$

$$1 - \tilde{\mathfrak{p}} = \frac{f_H m_{H0}}{f_L m_{L1} + f_H m_{H0} + f_H m_{H1}}$$

Equilibrium quantities

$$q_{1L} = m_{L1}(1-\alpha_e)$$

$$q_{1L}^{\pi} = m_{L1}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) = \frac{q_{1L}}{1-\alpha_e}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$

$$q_{1H} = m_{H0} + m_{H1}(1-\alpha_e)$$

$$q_{2H} = m_{H0}$$

$$q_{1H}^{\pi} = (m_{H0}+m_{H1})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$

$$\phi_0 = \beta$$
$$\phi_1 = \beta\sigma_L \frac{(1-\alpha_e)}{q_{1L}}$$

Type L has no incentives to hold $\mathfrak{p} = 0$-tokens whenever

$$\phi_0 > \beta\sigma_L \frac{(1-\pi)}{q_{1L}}(1-\alpha_e) + \beta\sigma_L \frac{\pi}{q_{1L}^{\pi}}(1-\alpha_e + \alpha_e(1-\tilde{\mathfrak{p}})) + \beta(1-\sigma_L)\varepsilon$$

$$1 > \frac{\sigma_L f_L m_{L1} + \sigma_H f_H m_{H1}}{f_L m_{L1} + f_H m_{H1}}(1-\alpha_e) + (1-\sigma_L)\varepsilon$$

Since $\frac{\sigma_L f_L m_{L1} + \sigma_H f_H m_{H1}}{f_L m_{L1} + f_H m_{H1}} \in (\sigma_H, \sigma_L)$, this condition holds whenever

$$1 > \sigma_L(1-\alpha_e) + (1-\sigma_L)\varepsilon,$$

which is always satisfied since $\alpha_e > 0$ and $\varepsilon < 1$. We now verify conditions such that, at the given prices, H-buyers want to acquire type $\mathfrak{p} = 1$-tokens. Suppose a H-buyer does not acquire $\mathfrak{p} = 1$-tokens, but only $\mathfrak{p} = 0$-tokens. Then she consumes

$$q_{1H} = m_{H0}$$
$$q_{2H} = m_{H0}$$
$$q_{1H}^{\pi} = m_{H0}(1-\alpha_e + \alpha_e(1-\tilde{\mathfrak{p}}))$$

where m_{H0} is given by

$$\phi_0 \quad = \quad \beta\sigma_H \frac{(1-\pi)}{m_{H0}} + \beta\sigma_H \frac{\pi}{m_{H0}} + \beta \frac{(1-\sigma_H)}{m_{H0}}$$

and so $m_{H0} = 1$. Then $q_{1H} = q_{2H} = 1$ and $q_{1H}^{\pi} = 1 - \alpha_e + \alpha_e(1-\tilde{\mathfrak{p}})$. The first order condition with respect to $\mathfrak{p} = 1$-tokens is

$$\begin{aligned} -\phi_1 + \beta\sigma_H \frac{(1-\pi)}{q_{1H}}(1-\alpha_e) + \beta\sigma_H \frac{\pi}{q_{1H}^{\pi}}(1-\alpha_e + \alpha_e(1-\tilde{\mathfrak{p}})) \quad &= \\ -\phi_1 + \beta\sigma_H(1-\pi)(1-\alpha_e) + \beta\sigma_H \pi & \end{aligned}$$

and since

$$\phi_1 \in (\beta(1-\alpha_e)\sigma_H, \beta(1-\alpha_e)\sigma_L)$$

the H-buyer will prefer to acquire $\mathfrak{p} = 1$-tokens if

$$\beta(1-\alpha_e)\sigma_L \quad < \quad \beta\sigma_H(1-\pi)(1-\alpha_e) + \beta\sigma_H\pi,$$

or,

$$\frac{(1-\alpha_e)(\sigma_L-\sigma_H)}{\sigma_H\alpha_e} \quad < \quad \pi.$$

This shows the following result,

Proposition 5. *When*

$$\pi \leq \bar{\pi} \equiv \frac{(\sigma_L-\sigma_H)(1-\alpha_e)}{\alpha_e\sigma_H},$$

the unique equilibrium is "separating": $\mathcal{M}_\mathrm{H}=\{0\}$,$\mathcal{M}_\mathrm{L}=\{1\}$. When $\pi>\bar{\pi}$, the unique equilibrium is "pooling": $\mathcal{M}_\mathrm{H}=\{0,1\}$,$\mathcal{M}_\mathrm{L}=\{1\}$.

The degree of singleness, $\mathcal{S}$ is given by

$$\mathcal{S}=\begin{cases}\pi, & \text{with programmable money}\\ 1, & \text{without programmable money}\end{cases} \tag{5}$$

and the degree of programmability is given by

$$\mathcal{P}=\begin{cases}\tilde{\mathfrak{p}}, & \text{with programmable money}\\ 0, & \text{without programmable money}\end{cases}$$

Proposition 6. *Imperfect recognizability increases programmability and singleness: a higher π leads to a higher degree of singleness $\mathcal{S}$,and a (weakly) higher degree of programmability $\mathcal{P}$.*

Proof. The first part is straightforward since (5) shows that $\mathcal{S}$ is increasing in π. For the second part, when $\pi<\bar{\pi}$, the equilibrium degree of programmability is

$$\tilde{\mathfrak{p}}=1-\frac{f_H}{\frac{f_L}{1-\alpha_e}+f_H}$$

which does not depend on π. Therefore, increasing π in this range does not increase programmability. When $\pi>\bar{\pi}$, the degree of programmability is

$$\tilde{\mathfrak{p}}=1-\frac{f_H m_{H0}}{f_L m_{L1}+f_H m_{H0}+f_H m_{H1}}$$

where

$$\begin{aligned}\phi_1 m_{H1} &= \frac{[\beta\sigma_L(1-\alpha_e)-\phi_1]}{[\phi_1-\beta(1-\alpha_e)\sigma_H]}\frac{\sigma_L f_L}{f_H}\beta\\ \phi_1 m_{H0} &= \phi_1\beta\frac{(1-\sigma_H)}{\beta-\phi_1}\\ \phi_1 m_{L1} &= \beta\sigma_L\end{aligned}$$

Hence

$$\tilde{\mathfrak{p}} = 1 - \frac{f_H \phi_1 \beta \frac{(1-\sigma_H)}{\beta-\phi_1}}{f_L \beta \sigma_L + f_H \phi_1 \beta \frac{(1-\sigma_H)}{\beta-\phi_1} + f_H \frac{[\beta\sigma_L(1-\alpha_e)-\phi_1]}{[\phi_1-\beta(1-\alpha_e)\sigma_H]} \frac{\sigma_L f_L}{f_H} \beta} = 1 - \frac{A(\phi_1)}{cst + A(\phi_1) + B(\phi_1)}$$

where $A'(\phi_1) > 0$ and $B'(\phi_1) < 0$. Hence

$$\begin{aligned} \tilde{\mathfrak{p}}'(\phi_1) &= -\frac{A'(\phi_1)\left[cst + A(\phi_1) + B(\phi_1)\right] - \left[A'(\phi_1) + B'(\phi_1)\right] A(\phi_1)}{\left[cst + A(\phi_1) + B(\phi_1)\right]^2} \\ &= -\frac{A'(\phi_1)\left[cst + B(\phi_1)\right] - B'(\phi_1) A(\phi_1)}{\left[cst + A(\phi_1) + B(\phi_1)\right]^2} < 0 \end{aligned}$$

where the inequality follows from $A'(\phi_1) > 0$ and $B'(\phi_1) < 0$. So, and intuitively, the degree of programmability is decreasing with the price of $\mathfrak{p} = 1$-tokens.

Next, ϕ_1 is implicitly given by (From FOC of buyer H wrt m_{H1})

$$\phi_1 = \beta\sigma_H \left[(1-\pi)\frac{(1-\alpha_e)}{m_{H0} + m_{H1}(1-\alpha_e)} + \pi \frac{1}{(m_{H0} + m_{H1})}\right] \tag{6}$$

So if π increases, more weight is given to uninformed-MU of cons $\frac{1}{(m_{H0}+m_{H1})}$ than to informed-MU of cons $\frac{(1-\alpha_e)}{m_{H0}+m_{H1}(1-\alpha_e)}$. At the same time, MUC-informed is lower than MUC-uninformed since $\alpha_e > 0$. Hence, when π increases, the expected MUC increases, holding everything else constant, the RHS increases (because more weight is given to the event with higher MUC) . This tends to increase ϕ_1. However, there is reshuffling of portfolio when ϕ_1 increases, since

$$\begin{aligned} m_{H0} &= \beta \frac{(1-\sigma_H)}{\beta - \phi_1} \\ m_{H1} &= \frac{1}{\phi_1} \frac{[\beta\sigma_L(1-\alpha_e) - \phi_1]}{[\phi_1 - \beta(1-\alpha_e)\sigma_H]} \frac{\sigma_L f_L}{f_H} \beta \end{aligned}$$

so m_{H0} increases with ϕ_1, but m_{H1} decreases with it. **If** the decline in m_{H1} is too pronounced, so that $m_{H0} + m_{H1}(1-\alpha_e)$ decreases fast with ϕ_1, then both MUC increases (by a lot) so that there are no increases in ϕ_1 that can re-establish the equality (after the increase in π). Then ϕ_1 has to drop (and not increase). In other words, if the elasticity of $q_{1H} = m_{H0} + m_{H1}(1-\alpha_e)$ with respect to ϕ_1 is less than -1, then the RHS will be amplifying the increase in ϕ_1, so that ϕ_1 should decrease rather than increase. Below we show that elasticity of $q_{1H} = m_{H0} + m_{H1}(1-\alpha_e)$ with respect to ϕ_1 is less than -1 is a sufficient condition for $d\phi_1/d\pi < 0$.

The first step is to re-arrange (6),

$$\phi_1 m_{H0} + \phi_1 m_{H1}(1-\alpha_e) = \beta\sigma_H(1-\alpha_e) + \beta\sigma_H \pi \alpha_e \frac{\phi_1 m_{H0}}{(\phi_1 m_{H0} + \phi_1 m_{H1})}$$

or

$$
\begin{aligned}
\phi_1 \frac{(1-\sigma_H)}{\beta-\phi_1} + \frac{[\beta\sigma_L(1-\alpha_e)-\phi_1]}{[\phi_1-\beta(1-\alpha_e)\sigma_H]}\frac{\sigma_L f_L}{f_H}(1-\alpha_e) &= \sigma_H(1-\alpha_e) + \sigma_H \pi \alpha_e \frac{\phi_1\beta\frac{(1-\sigma_H)}{\beta-\phi_1}}{\left[\phi_1\beta\frac{(1-\sigma_H)}{\beta-\phi_1} + \frac{[\beta\sigma_L(1-\alpha_e)-\phi_1]}{[\phi_1-\beta(1-\alpha_e)\sigma_H]}\frac{\sigma_L f_L}{f_H}\beta\right]} \\
\frac{1}{f_H}A(\phi_1) + \frac{1}{f_H}B(\phi_1)(1-\alpha_e) &= \sigma_H(1-\alpha_e) + \sigma_H\pi\alpha_e \frac{\frac{1}{f_H}A(\phi_1)}{\left[\frac{1}{f_H}A(\phi_1) + \frac{1}{f_H}B(\phi_1)\right]} \\
A(\phi_1) + B(\phi_1)(1-\alpha_e) &= f_H\sigma_H(1-\alpha_e) + \pi f_H \sigma_H \alpha_e \frac{A(\phi_1)}{[A(\phi_1)+B(\phi_1)]}
\end{aligned}
$$

($A(\phi_1) = f_H\phi_1 m_{H0}$, and $B(\phi_1) = \frac{f_H}{1-\alpha_e}\phi_1 m_{H1}$) Hence,

$$
\begin{aligned}
[A'(\phi_1) + B'(\phi_1)(1-\alpha_e)]\, d\phi_1 &= f_H\sigma_H\alpha_e \frac{A(\phi_1)}{[A(\phi_1)+B(\phi_1)]} d\pi + \pi f_H\sigma_H\alpha_e \frac{A'(\phi_1)\,[A(\phi_1)+B(\phi_1)] - A(\phi_1)\,[A'(\phi_1)+B'(\cdots}{[A(\phi_1)+B(\phi_1)]^2} \\
[A'(\phi_1) + B'(\phi_1)(1-\alpha_e)]\, d\phi_1 &= f_H\sigma_H\alpha_e \frac{A(\phi_1)}{[A(\phi_1)+B(\phi_1)]} d\pi + \pi f_H\sigma_H\alpha_e \frac{A'(\phi_1)B(\phi_1) - A(\phi_1)B'(\phi_1)}{[A(\phi_1)+B(\phi_1)]^2} d\phi_1
\end{aligned}
$$

Therefore

$$
\frac{d\phi_1}{d\pi} = f_H\sigma_H\alpha_e \frac{\frac{A(\phi_1)}{[A(\phi_1)+B(\phi_1)]}}{\left[A'(\phi_1) + B'(\phi_1)(1-\alpha_e) - f_H\sigma_H\alpha_e\pi\frac{B(\phi_1)A'(\phi_1) - B'(\phi_1)A(\phi_1)}{[A(\phi_1)+B(\phi_1)]^2}\right]}
$$

Since $A'(\phi_1) > 0$ and $B'(\phi_1) < 0$, the sign of the denominator is not clear. But a sufficient condition for it to be negative is $A'(\phi_1) + B'(\phi_1)(1-\alpha_e) \leq 0$, or ($\phi_1 m_{H1}$ more responsive than $\phi_1 m_{H0}$)

$$
\begin{aligned}
A(\phi_1) &= f_H\phi_1\beta\frac{(1-\sigma_H)}{\beta-\phi_1} \\
A'(\phi_1) &= f_H\beta\frac{(1-\sigma_H)}{\beta-\phi_1} + f_H\phi_1\beta\frac{(1-\sigma_H)}{(\beta-\phi_1)^2} \\
A'(\phi_1) &= f_H\beta\frac{(1-\sigma_H)(\beta-\phi_1) + \phi_1(1-\sigma_H)}{(\beta-\phi_1)^2} \\
A'(\phi_1) &= f_H\beta^2\frac{(1-\sigma_H)}{(\beta-\phi_1)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
B(\phi_1) &= f_H\frac{[\beta\sigma_L(1-\alpha_e)-\phi_1]}{[\phi_1-\beta(1-\alpha_e)\sigma_H]}\frac{\sigma_L f_L}{f_H}\beta \\
B'(\phi_1) &= \sigma_L f_L\beta\frac{-[\phi_1-\beta(1-\alpha_e)\sigma_H] - [\beta\sigma_L(1-\alpha_e)-\phi_1]}{[\phi_1-\beta(1-\alpha_e)\sigma_H]^2} \\
B'(\phi_1) &= -\sigma_L f_L\beta^2\frac{(1-\alpha_e)(\sigma_L-\sigma_H)}{[\phi_1-\beta(1-\alpha_e)\sigma_H]^2}
\end{aligned}
$$

Therefore,

$$
A'(\phi_1) + B'(\phi_1)(1-\alpha_e) = f_H\beta^2\frac{(1-\sigma_H)}{(\beta-\phi_1)^2} - \sigma_L f_L\beta^2\frac{(1-\alpha_e)^2(\sigma_L-\sigma_H)}{[\phi_1-\beta(1-\alpha_e)\sigma_H]^2}
$$

This is negative whenever

$$\begin{aligned} f_H \beta^2 \frac{(1-\sigma_H)}{(\beta-\phi_1)^2} &\leq \sigma_L f_L \beta^2 \frac{(1-\alpha_e)^2 (\sigma_L-\sigma_H)}{[\phi_1 - \beta(1-\alpha_e)\sigma_H]^2} \\ \frac{[\phi_1 - \beta(1-\alpha_e)\sigma_H]^2}{(\beta-\phi_1)^2} &\leq \frac{\sigma_L f_L}{f_H}(1-\alpha_e)^2 \frac{(\sigma_L-\sigma_H)}{(1-\sigma_H)} \end{aligned}$$

Since $\phi_1 \in (\beta\sigma_H(1-\alpha_e), \beta\sigma_L(1-\alpha_e))$ and the LHS is increasing in ϕ_1, a sufficient condition is

$$\begin{aligned} \frac{[\beta\sigma_L(1-\alpha_e) - \beta(1-\alpha_e)\sigma_H]^2}{(\beta - \beta\sigma_L(1-\alpha_e))^2} &\leq \frac{\sigma_L f_L}{f_H}(1-\alpha_e)^2 \frac{(\sigma_L-\sigma_H)}{(1-\sigma_H)} \\ \frac{\beta^2 (\sigma_L-\sigma_H)^2 (1-\alpha_e)^2}{\beta^2(1-\sigma_L(1-\alpha_e))^2} &\leq \frac{\sigma_L f_L}{f_H}(1-\alpha_e)^2 \frac{(\sigma_L-\sigma_H)}{(1-\sigma_H)} \\ \frac{(\sigma_L-\sigma_H)}{(1-\sigma_L(1-\alpha_e))^2} &\leq \frac{\sigma_L f_L}{f_H} \frac{1}{(1-\sigma_H)} \end{aligned}$$

Then $d\phi_1/d\pi < 0$ and then $\tilde{\mathfrak{p}}$ is increasing in π in the equilibrium where $\pi > \bar{\pi}$. □

Numerical Example

Comparative statics wrt α_e: For $\alpha_e \in (0, 0.095)$, case i is an equilibrium, for $\alpha_e \in (0.095, 1)$, case iv is an equilibrium.

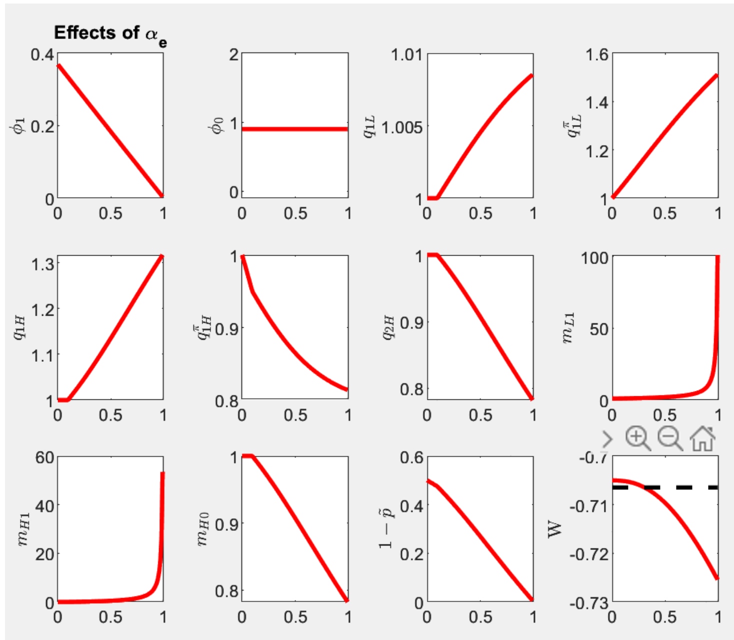

Figure 2: Effects of α_e (dash: welfare without programmability)

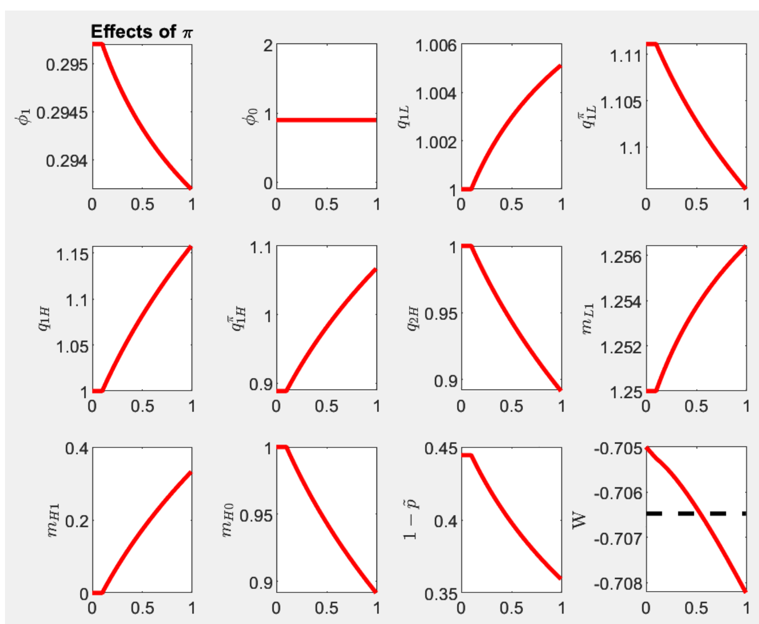

Figure 3: Effects of π (dash: welfare without programmability)

Appendix

Proof of Proposition 1:

In equilibrium, there are potentially two tokens in circulation: $\mathfrak{p}_1 = 0$ and $\mathfrak{p}_0 = 1$. We use $\mathcal{M}_\mathrm{i}$ to denote the set of tokens chosen by type i. There are potentially nine equilibrium outcomes.

			$\mathcal{M}_\mathrm{L}$	
		{1}	{0}	{0,1}
	{1}	x	x	x
$\mathcal{M}_\mathrm{H}$	{0}	i	ii	iii
	{0,1}	iv	v	vi

Obviously, type H holding only $\mathfrak{p}_1$ is not an equilibrium because $q_{2H} = 0$ and, for any finite ϕ_0, type H have an incentive to hold some $\mathfrak{p}_0$. If there were initially no $\mathfrak{p}_0$ tokens, bankers can make a profit by creating some. Hence, we only need to consider six remaining cases.

Case (i): $\mathcal{M}_\mathrm{H} = \{0\}, \mathcal{M}_\mathrm{L} = \{1\}$

The equilibrium conditions for the L type, H type and the banker are given by:

$$\phi_1 = \beta\sigma_L u'(q_{1L})(1 - \alpha_e)$$

$$\phi_0 = \beta\sigma_H u'(q_{1H}) + \beta(1 - \sigma_H)u'(q_{2H})$$

$$\phi_0 = \beta$$

$$\phi_1 = \beta\sigma_L(1 - \alpha_e)$$

Since $q_{1H} = q_{2H} \equiv q_H$, we have

$$\phi_0 = \beta u'(q_H)$$

or

$$u'(q_H) = 1.$$

Type H has no incentive to hold $\mathfrak{p}_1$ if

$$\phi_1 > \beta\sigma_H u'(q_H)(1 - \alpha_e)$$

$$\sigma_L > \sigma_H.$$

and type L has not incentives to hold $\mathfrak{p}_0$if

$$\phi_0 > \beta\sigma_L u'(q_{1L}) + \beta(1 - \sigma_L)\varepsilon$$

or

$$1 > \varepsilon.$$

So this is an equilibrium.

Case (ii): $\mathcal{M}_{\mathrm{H}} = \{0\}$,$\mathcal{M}_{\mathrm{L}} = \{0\}$

The equilibrium conditions for the L type, H type and the banker are given by:

$$\begin{aligned}
\phi_0 &= \beta\sigma_L u'(q_{1L}) + \beta(1-\sigma_L)\varepsilon \\
\phi_0 &= \beta\sigma_H u'(q_{1H}) + \beta(1-\sigma_H)u'(q_{2H}) \\
\phi_0 &= \beta \\
q_{1H} &= q_{2H} \equiv q_H
\end{aligned}$$

Hence we have

$$1 = u'(q_H) = \sigma_L u'(q_{1L}) + (1-\sigma_L)\varepsilon.$$

We now check the incentives to offer $\mathfrak{p}_1$ to serve type L only. Type L has an incentive to hold if

$$\begin{aligned}
&\phi_1 \leq \beta\sigma_L u'(q_{1L})(1-\alpha_e) \\
\Rightarrow &\phi_1 \leq \beta(1-(1-\sigma_L)\varepsilon)(1-\alpha_e).
\end{aligned}$$

Type H has no incentives to hold if

$$\phi_1 > \beta\sigma_H u'(q_H)(1-\alpha_e) = \beta\sigma_H(1-\alpha_e)$$

And the banker makes non-zero profit if

$$\phi_1 > \beta\sigma_L(1-\alpha_e)$$

Since $\sigma_L > \sigma_H$, it is profitable to introduce $\mathfrak{p}_1$ iff

$$1-(1-\sigma_L)\varepsilon > \sigma_L$$

or

$$1 > \varepsilon.$$

Since the proposed equilibrium can be disturbed, this is not an equilibrium.

Case (iii) :$\mathcal{M}_{\mathrm{H}} = \{0\}$,$\mathcal{M}_{\mathrm{L}} = \{0, 1\}$

The equilibrium conditions for the L type, H type and the banker are given by:

$$\phi_1 = \beta\sigma_L u'(q_{1L})(1-\alpha_e)$$
$$\phi_0 = \beta\sigma_L u'(q_{1L}) + \beta(1-\sigma_L)\varepsilon$$
$$\phi_0 = \beta\sigma_H u'(q_{1H}) + \beta(1-\sigma_H)u'(q_{2H})$$
$$\phi_0 = \beta$$
$$\phi_1 = \beta\sigma_L(1-\alpha_e)$$

These imply that

$$\beta = \beta\sigma_L + \beta(1-\sigma_L)\varepsilon$$

or

$$1 = \sigma_L + (1-\sigma_L)\varepsilon$$

which contracts with the assumption that $\varepsilon < 1$. **So this is not an equilibrium.**

Case (iv) :$\mathcal{M}_{\mathrm{H}} = \{0,1\}$,$\mathcal{M}_{\mathrm{L}} = \{1\}$

The equilibrium conditions for the L type, H type and the banker are given by:

$$\phi_1 = \beta\sigma_L u'(q_{1L})(1-\alpha_e)$$
$$\phi_1 = \beta\sigma_H u'(q_{1H})(1-\alpha_e)$$
$$\phi_0 = \beta\sigma_H u'(q_{1H}) + \beta(1-\sigma_H)u'(q_{2H}) = \frac{\phi_1}{(1-\alpha_e)} + \beta(1-\sigma_H)u'(q_{2H})$$
$$\phi_0 = \beta$$
$$\phi_1 = \beta\frac{f_L m_L \sigma_L + f_H m_H \sigma_H}{f_L m_L + f_H m_H}(1-\alpha_e)$$

Note that the last condition implies that

$$\phi_1 > \beta\sigma_H(1-\alpha_e),$$

which, together with the second FOC above, imply that

$$u'(q_{1H}) > 1.$$

But the third FOC above also implies that

$$\beta = \frac{\phi_1}{(1-\alpha_e)} + \beta(1-\sigma_H)u'(q_{2H}) > \beta\sigma_H + \beta(1-\sigma_H)u'(q_{2H})$$

implying

$$u'(q_{2H}) < 1.$$

This contradicts with the fact that

$$q_{1H} \geq q_{2H}.$$

So this is not an equilibrium.

Case (v) :$\mathcal{M}_{\mathrm{H}} = \{0,1\}$,$\mathcal{M}_{\mathrm{L}} = \{0\}$

The equilibrium conditions for the L type, H type and the banker are given by:

$$\begin{aligned}
\phi_0 &= \beta\sigma_L u'(q_{1L}) + \beta(1-\sigma_L)\varepsilon \\
\phi_1 &= \beta\sigma_H u'(q_{1H})(1-\alpha_e) \\
\phi_0 &= \beta\sigma_H u'(q_{1H}) + \beta(1-\sigma_H)u'(q_{2H}) = \frac{\phi_1}{(1-\alpha_e)} + \beta(1-\sigma_H)u'(q_{2H}) \\
\phi_0 &= \beta \\
\phi_1 &= \beta\sigma_H(1-\alpha_e)
\end{aligned}$$

The last three conditions imply that

$$\phi_0 = \beta = \beta\sigma_H + \beta(1-\sigma_H)u'(q_{2H})$$

implying

$$u'(q_{2H}) = 1.$$

The first FOC implies that

$$1 = \sigma_L u'(q_{1L}) + (1-\sigma_L)\varepsilon$$

implying that

$$u'(q_{1L}) > 1$$

Finally, the fact that type L does not hold $\mathfrak{p}_1$ requires that

$$\phi_1 > \beta\sigma_L u'(q_{1L})(1-\alpha_e)$$

or

$$1 > \frac{\sigma_H}{\sigma_L} > u'(q_{1L})$$

but this contradicts with the condition above. **So this is not an equilibrium.**

Case (vi) :$\mathcal{M}_{\mathrm{H}} = \{0,1\}$,$\mathcal{M}_{\mathrm{L}} = \{0,1\}$

The equilibrium conditions for the L type, H type and the banker are given by:

$$\phi_1 = \beta\sigma_L u'(q_{1L})(1-\alpha_e)$$
$$\phi_0 = \beta\sigma_L u'(q_{1L}) + \beta(1-\sigma_L)\varepsilon$$
$$\phi_1 = \beta\sigma_H u'(q_{1H})(1-\alpha_e)$$
$$\phi_0 = \beta\sigma_H u'(q_{1H}) + \beta(1-\sigma_H)u'(q_{2H})$$
$$\phi_0 = \beta$$
$$\phi_1 = \beta\frac{f_L m_L \sigma_L + f_H m_H \sigma_H}{f_L m_L + f_H m_H}(1-\alpha_e)$$

The first four conditions imply that

$$\phi_0 = \phi_1/(1-\alpha_e) + \beta(1-\sigma_L)\varepsilon$$
$$\phi_0 = \phi_1/(1-\alpha_e) + \beta(1-\sigma_H)u'(q_{2H})$$

implying

$$1 > \frac{1-\sigma_L}{1-\sigma_H}\varepsilon = u'(q_{2H})$$

which then implies that

$$1 > u'(q_{2H}) \geq u'(q_{1H})$$

but this contradicts with the condition that

$$\beta = \phi_0 = \beta\sigma_H u'(q_{1H}) + \beta(1-\sigma_H)u'(q_{2H}).$$

So this is not an equilibrium.

To prove: Case (i) and (iv) re equilibria with CRRA preferences:

CASE (i) : Consider an equilibrium with $\mathcal{M}_{\mathbf{H}} = \{0\}$,$\mathcal{M}_{\mathbf{L}} = \{1\}$

The consumption levels are then

$$\begin{aligned} q_{1H} &= m_{H0} \\ q_{1H}^{\pi} &= m_{H0}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\ q_{2H} &= m_{H0} \\ q_{1L} &= m_{L1}(1-\alpha_e) \\ q_{1L}^{\pi} &= m_{L1}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\ q_{2L} &= 0 \end{aligned}$$

The equilibrium conditions for the L type, H type and the banker are given by:

$$\phi_1 = \beta\sigma_L u'(q_{1L})(1-\pi)(1-\alpha_e) + \beta\sigma_L u'(q_{1L}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$

$$\phi_0 = \beta\sigma_H u'(q_{1H})(1-\pi) + \beta\sigma_H u'(q_{1H}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta(1-\sigma_H)u'(q_{2H})$$

$$\phi_0 = \beta$$

$$\phi_1 = \beta\sigma_L(1-\alpha_e)$$

$$1-\tilde{\mathfrak{p}} = \frac{f_H m_{H0}}{f_L m_{L1} + f_H m_{H0}}$$

Since $m_{H0} = q_{1H}$ and $(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))\, m_{H0} = q_{1H}^{\pi}$ we have

$$q_{1H} = \frac{q_{1H}^{\pi}}{1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})}$$

Since $q_{1H} = q_{2H} \equiv q_H$, we have

$$q_{1H} = q_{2H} = \frac{q_{1H}^{\pi}}{1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})}$$

First, type L has no incentives to hold $\mathfrak{p}_0$:

$$\phi_0 > \beta\sigma_L u'(q_{1L})(1-\pi) + \beta\sigma_L u'(q_{1L}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta(1-\sigma_L)\varepsilon$$

or

$$1 > \sigma_L u'(q_{1L})(1-\pi) + \sigma_L u'(q_{1L}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + (1-\sigma_L)\varepsilon$$

Note that the FOC of the low type given ϕ_1 implies that (setting $\mathfrak{p}=0$):

$$\begin{aligned} -\phi_{\mathfrak{p}} + \beta\sigma_L u'(q_{1L})(1-\pi)(1-\alpha_e) + \beta\sigma_L u'(q_{1L}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) &= 0 \\ -\sigma_L(1-\alpha_e) + \sigma_L u'(q_{1L})(1-\pi)(1-\alpha_e) + \sigma_L u'(q_{1L}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) &= 0 \end{aligned}$$

so

$$\sigma_L u'(q_{1L})(1-\pi)(1-\alpha_e) + \sigma_L u'(q_{1L}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) = \sigma_L(1-\alpha_e)$$

$$\begin{aligned} &\sigma_L u'(q_{1L})(1-\pi) + \sigma_L u'(q_{1L}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\ =&\sigma_L(1-\alpha_e) + \alpha_e\sigma_L u'(q_{1L})(1-\pi). \end{aligned}$$

Using this result, the above condition becomes

$$1 > \sigma_L(1-\alpha_e) + \alpha_e\sigma_L u'(q_{1L})(1-\pi) + (1-\sigma_L)\varepsilon$$

Also, from the low type's FOC, we know that

$$\begin{aligned} \beta\sigma_L(1-\alpha_e) &> \beta\sigma_L u'(q_{1L})(1-\pi)(1-\alpha_e) \\ &\Rightarrow 1 > u'(q_{1L})(1-\pi) \end{aligned}$$

Therefore the RHS of the above condition is

$$\begin{aligned} &\sigma_L(1-\alpha_e) + \alpha_e\sigma_L u'(q_{1L})(1-\pi) + (1-\sigma_L)\varepsilon \\ <&\sigma_L(1-\alpha_e) + \alpha_e\sigma_L + (1-\sigma_L)\varepsilon \\ =&\sigma_L + (1-\sigma_L)\varepsilon \\ <&1. \end{aligned}$$

Hence the low type has no incentives to hold $\mathfrak{p}_0$.

Next, type H has no incentives to hold $\mathfrak{p}_1$ if

$$\phi_1 > \beta\sigma_H u'(q_{1H})(1-\pi)(1-\alpha_e) + \beta\sigma_H u'(q_{1H}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})).$$

This requires

$$\begin{aligned} \beta\sigma_L(1-\alpha_e) &> \underbrace{\beta\sigma_H u'(q_{1H})(1-\pi)(1-\alpha_e) + \beta\sigma_H u'(q_{1H}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})).}_{=\phi_0-\beta(1-\sigma_H)u'(q_{2H})} \\ \sigma_L(1-\alpha_e) + (1-\sigma_H)u'(q_{2H}) &> 1 \end{aligned}$$

Notice that (FOC H):

$$\begin{aligned} 1 &= u'(q_{2H}) - \pi\sigma_H\left[u'(q_{2H}) - u'(q_{1H}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))\right] \\ 1 &= u'(q_{2H}) - \pi\sigma_H\left[u'(q_{2H}) - u'(q_{2H}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})))(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))\right] \\ 1 &= u'(q_{2H}) - \frac{\pi\sigma_H}{q_{2H}}\left[u'(q_{2H})q_{2H} - u'(q_{2H}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})))q_{2H}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))\right] \end{aligned}$$

$$
\begin{aligned}
1 &= u'(q_{1H}) - \pi\sigma_H \left[u'(q_{2H}) - u'(q_{1H}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))\right] \\
1 &= u'(q_{1H}) \left[1-\pi\sigma_H + \pi\sigma_H(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]
\end{aligned}
$$

Since u is concave, $u''(q) < 0$ for all q. Suppose the coefficient of relative risk aversion is less than 1, then $u'(x)x$ is increasing. Then $u'(\alpha q)\alpha q < u'(q)q$ for all $\alpha < 1$. In this case, (or in the homothetic case)

$$
u'(q_{2H}) \geq 1.
$$

Hence H has no incentive to hold $\mathfrak{p}_1$ whenever

$$
\begin{aligned}
\beta\sigma_L(1-\alpha_e) &> \beta\sigma_H u'(q_{1H}) \left[(1-\pi)(1-\alpha_e) + \pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]. \\
\frac{\sigma_L(1-\alpha_e)}{\sigma_H} &> \frac{\left[(1-\pi)(1-\alpha_e) + \pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]}{\left[1-\pi\sigma_H+\pi\sigma_H(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]}.
\end{aligned}
$$

Since $\tilde{\mathfrak{p}} < 1$ and does not go to one as $\alpha_e \to 1$ (if anything $\tilde{\mathfrak{p}} \to 0$ in that case) the inequality above shows the conjecture that for α_e sufficiently high, this equilibrium no longer exists because ϕ_1 becomes so low that the H type chooses to hold some $\mathfrak{p}_1$ token.

CASE (ii) : Consider an equilibrium with $\mathcal{M}_{\mathbf{H}} = \{0\}$,$\mathcal{M}_{\mathbf{L}} = \{0\}$

The consumption levels are then

$$
\begin{aligned}
q_{1H} &= q_{1H}^{\pi} = q_{2H} = m_{H0} \\
q_{1L} &= q_{1L}^{\pi} = q_{2L} = m_{L0}
\end{aligned}
$$

The equilibrium conditions for the L type, H type and the banker are given by:

$$
\begin{aligned}
&\phi_0 = \beta\sigma_L u'(q_{1L}) + \beta(1-\sigma_L)\varepsilon \\
&\phi_0 = \beta u'(q_{1H}) \\
&\phi_0 = \beta \\
&\tilde{\mathfrak{p}} = 0
\end{aligned}
$$

We now check the incentives to offer $\mathfrak{p}_1$ to serve type L only. Type L has an incentive to hold if

$$
\phi_1 \leq \beta\sigma_L u'(q_{1L})(1-\pi)(1-\alpha_e) + \beta\sigma_L u'(q_{1L})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))
$$

So

$$
\begin{aligned}
\phi_1 &\leq \beta\sigma_L u'(q_{1L}) \left[(1-\alpha_e) + \pi\alpha_e(1-\tilde{\mathfrak{p}})\right] \\
\Rightarrow \phi_1 &\leq \beta(1-(1-\sigma_L)\varepsilon) \left[(1-\alpha_e) + \pi\alpha_e(1-\tilde{\mathfrak{p}})\right]
\end{aligned}
$$

Type H has no incentives to hold if

$$\begin{aligned}\phi_1 &> \beta\sigma_H u'(q_{1H})(1-\pi)(1-\alpha_e)+\beta\sigma_H u'(q_{1H})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\ &> \beta\sigma_H u'(q_{1H})\left[(1-\alpha_e)+\pi\alpha_e(1-\tilde{\mathfrak{p}}))\right]\end{aligned}$$

And the banker makes non-zero profit if

$$\phi_1 > \beta\sigma_L(1-\alpha_e)$$

It is profitable to introduce $\mathfrak{p}_1$ iff

$$\begin{aligned}\beta(1-(1-\sigma_L)\varepsilon)\left[(1-\alpha_e)+\pi\alpha_e(1-\tilde{\mathfrak{p}})\right] &> \beta\sigma_L(1-\alpha_e) \\ (1-\varepsilon+\sigma_L\varepsilon)\left[(1-\alpha_e)+\pi\alpha_e(1-\tilde{\mathfrak{p}})\right] &> \sigma_L(1-\alpha_e)\end{aligned}$$

in the worst case scenario $\tilde{\mathfrak{p}}=1$ then it is profitable to introduce $\mathfrak{p}_1$ whenever

$$1 \geq \varepsilon,$$

which is always the case. **So $\mathcal{M}_{\mathbf{H}}=\{0\}$,$\mathcal{M}_{\mathbf{L}}=\{0\}$ cannot be an equilibrium.**

CASE (iii): Consider an equilibrium with $\mathcal{M}_{\mathbf{H}}=\{0\}$,$\mathcal{M}_{\mathbf{L}}=\{0,1\}$

The consumption levels are then

$$\begin{aligned}q_{1H} &= m_{H0} \\ q_{1H}^{\pi} &= m_{H0}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\ q_{2H} &= m_{H0} \\ q_{1L} &= m_{L0}+m_{L1}(1-\alpha_e) \\ q_{1L}^{\pi} &= (m_{L0}+m_{L1})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\ q_{2L} &= m_{L0}\end{aligned}$$

The equilibrium conditions for the L type, H type and the banker are given by:

$$
\begin{aligned}
\phi_0 &= \beta\sigma_L(1-\pi)u'(q_{1L}) + \beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta(1-\sigma_L)\varepsilon \\
\phi_1 &= \beta\sigma_L(1-\pi)u'(q_{1L})(1-\alpha_e) + \beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\
\phi_0 &= \beta\sigma_H(1-\pi)u'(q_{1H}) + \beta\sigma_H u'(q_{1H}^{\pi})\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta(1-\sigma_H)u'(q_{2H}) \\
\phi_0 &= \beta \\
\phi_1 &= \beta\sigma_L(1-\alpha_e) \\
1-\tilde{\mathfrak{p}} &= \frac{f_H m_{H0} + f_L m_{L0}}{f_L m_{L1} + f_L m_{L0} + f_H m_{H0}} > 0
\end{aligned}
$$

Notice (important for below) that the second FOC implies

$$1 > (1-\pi)u'(q_{1L})$$

This implies

$$
\begin{aligned}
1 &= \sigma_L\left[(1-\pi)u'(q_{1L}) + \pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))\right] + (1-\sigma_L)\varepsilon \\
\sigma_L(1-\alpha_e) &= \sigma_L\left[(1-\pi)u'(q_{1L}) + \pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))\right] - \sigma_L(1-\pi)u'(q_{1L})\alpha_e
\end{aligned}
$$

and subtracting both equations,

$$
\begin{aligned}
1-\sigma_L(1-\alpha_e) &= (1-\sigma_L)\varepsilon + \sigma_L(1-\pi)u'(q_{1L})\alpha_e \\
1-\sigma_L+\sigma_L\alpha_e &= (1-\sigma_L)\varepsilon + \sigma_L(1-\pi)u'(q_{1L})\alpha_e
\end{aligned}
$$

Since $1 > (1-\pi)u'(q_{1L})$, this contradicts $\varepsilon < 1$. **So $\mathcal{M}_{\mathbf{H}} = \{0\}$,$\mathcal{M}_{\mathbf{L}} = \{0,1\}$ cannot be an equilibrium.**

CASE (iv): Consider an equilibrium with $\mathcal{M}_{\mathbf{H}} = \{0,1\}$,$\mathcal{M}_{\mathbf{L}} = \{1\}$

In this case, the consumption levels are

$$
\begin{aligned}
q_{1L} &= m_{L1}(1-\alpha_e) \\
q_{1L}^{\pi} &= m_{L1}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\
q_{2L} &= 0
\end{aligned}
$$

and

$$
\begin{aligned}
q_{1H} &= m_{H0} + m_{H1}(1-\alpha_e) \geq (m_{H0}+m_{H1})(1-\alpha_e) \equiv \tilde{q}_H \\
q_{1H}^{\pi} &= (m_{H0}+m_{H1})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\
q_{2H} &= m_{H0} \leq q_{1H}
\end{aligned}
$$

The equilibrium conditions are

$$\phi_1 = \beta\sigma_L(1-\pi)u'(q_{1L})(1-\alpha_e) + \beta\sigma_L\pi u'(q_{1L}^\pi)(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$

$$\phi_1 = \beta\sigma_H(1-\pi)u'(q_{1H})(1-\alpha_e) + \beta\sigma_H\pi u'(q_{1H}^\pi)(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$

$$\phi_0 > \beta\sigma_L(1-\pi)u'(q_{1L}) + \beta\sigma_L\pi u'(q_{1L}^\pi)(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta(1-\sigma_L)\varepsilon$$

$$\phi_0 = \beta\sigma_H(1-\pi)u'(q_{1H}) + \beta\sigma_H\pi u'(q_{1H}^\pi)(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta(1-\sigma_H)u'(q_{2H})$$

$$\phi_0 = \beta$$

$$\phi_1 = \beta\frac{\sigma_L\left(f_Lm_{L1}\right)+\sigma_H\left(f_Hm_{H1}\right)}{f_Lm_{L1}+f_Hm_{H1}}(1-\alpha_e) = \beta\tilde{\phi}(1-\alpha_e)$$

$$1-\tilde{\mathfrak{p}} = \frac{f_Hm_{H0}}{f_Lm_{L1}+f_Hm_{H0}+f_Hm_{H1}} < 1$$

This is an equilibrium condition whenever at $m_{H1}=0$, the H buyer wants to purchase $\mathfrak{p}_1$. Define

$$\begin{aligned}\tilde{q}_{1H} &= m_{H0}\\ \tilde{q}_{1H}^\pi &= m_{H0}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))\\ q_{2H} &= m_{H0}\end{aligned}$$

Then H buyer wants to buy $\mathfrak{p}_1$ iff

$$\begin{aligned}\phi_1 &< \beta\sigma_H(1-\pi)u'(\tilde{q}_{1H})(1-\alpha_e) + \beta\sigma_H\pi u'(\tilde{q}_{1H}^\pi)(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))\\ \phi_1 &< \beta\sigma_H(1-\pi)u'(q_{2H})(1-\alpha_e) + \beta\sigma_H\pi u'(q_{2H})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\\ \phi_1 &< \left[\beta\sigma_H(1-\pi)(1-\alpha_e) + \beta\sigma_H\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]u'(q_{2H})\end{aligned}$$

and we also have

$$\begin{aligned}\phi_0 &\leq \beta\sigma_H(1-\pi)u'(q_{2H}) + \beta\sigma_H\pi u'(q_{2H})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k + \beta(1-\sigma_H)u'(q_{2H})\\ 1 &\leq \left[1-\sigma_H\pi+\sigma_H\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]u'(q_{2H})\end{aligned}$$

Hence a necessary condition is

$$\begin{aligned}\phi_1 &< \frac{\left[\beta\sigma_H(1-\pi)(1-\alpha_e) + \beta\sigma_H\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]}{[1-\sigma_H\pi+\sigma_H\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k]}\\ \tilde{\phi}(1-\alpha_e) &< \frac{\sigma_H(1-\pi)(1-\alpha_e)+\sigma_H\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k}{[1-\sigma_H\pi+\sigma_H\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k]}\end{aligned}$$

At the same time, it must be that L-buyers do not want to purchase $\mathfrak{p}_0$. Using the FOC with respect to $\mathfrak{p}_1$, since

$$u'(q_{1L}) = \frac{\tilde{\phi}(1-\alpha_e)}{\sigma_L(1-\pi)(1-\alpha_e)+\sigma_L\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k}$$

The FOC wrt $\mathfrak{p}_0$ implies,

$$1 \quad > \quad \frac{(1-\pi)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k}{(1-\pi)(1-\alpha_e)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k}\tilde{\phi}(1-\alpha_e)+(1-\sigma_L)\varepsilon.$$

Hence, $\mathcal{M}_{\mathbf{H}}=\{0,1\}$,$\mathcal{M}_{\mathbf{L}}=\{1\}$ **is an equilibrium whenever**

$$\tilde{\phi}(1-\alpha_e)<[1-(1-\sigma_L)\varepsilon]\,\frac{(1-\pi)(1-\alpha_e)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k}{(1-\pi)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k},$$

and

$$\tilde{\phi}(1-\alpha_e)<\sigma_H\frac{(1-\pi)(1-\alpha_e)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k}{[1-\sigma_H\pi+\sigma_H\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k]}$$

Which is the tighter upper-bound on $\tilde{\phi}(1-\alpha_e)$?

$$\begin{aligned}
[1-(1-\sigma_L)\varepsilon]\,\frac{(1-\pi)(1-\alpha_e)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k}{(1-\pi)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k} &< \sigma_H\frac{(1-\pi)(1-\alpha_e)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k}{[1-\sigma_H\pi+\sigma_H\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k]} \\
[1-(1-\sigma_L)\varepsilon]\left[1-\sigma_H\pi+\sigma_H\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right] &< \sigma_H\left[(1-\pi)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right] \\
1-\sigma_H-(1-\sigma_L)\varepsilon\left[1-\sigma_H\pi+\sigma_H\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right] &< 0
\end{aligned}$$

or

$$\frac{1-\sigma_H}{1-\sigma_L}<\varepsilon\left[1-\sigma_H\pi+\sigma_H\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]$$

which cannot be. So the tighter constraint is

$$\tilde{\phi}(1-\alpha_e)<\sigma_H\frac{(1-\pi)(1-\alpha_e)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k}{[1-\sigma_H\pi+\sigma_H\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k]}.$$

Therefore that bounds is separating case (i) and (iv).

CASE (v) : Next consider **the equilibrium with** $\mathcal{M}_{\mathbf{H}}=\{0,1\}$,$\mathcal{M}_{\mathbf{L}}=\{0\}$

In this case, the consumption levels are

$$\begin{aligned}
q_{1L} &= m_{L0} \\
q_{1L}^{\pi} &= m_{L0}(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))\leq q_{1L} \\
q_{2L} &= m_{L0}
\end{aligned}$$

and

$$\begin{aligned}
q_{1H} &= m_{H0}+m_{H1}(1-\alpha_e) \\
q_{1H}^{\pi} &= (m_{H0}+m_{H1})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\
q_{2H} &= m_{H0}\leq q_{1H}
\end{aligned}$$

We want to show that this cannot be an equilibrium because L buyers would want to purchase $\mathfrak{p}_1$.

The equilibrium conditions are

$$\phi_0 = \beta\sigma_L(1-\pi)u'(q_{1L}) + \beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta(1-\sigma_L)\varepsilon$$
$$\phi_1 > \beta\sigma_L(1-\pi)u'(q_{1L})(1-\alpha_e) + \beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$
$$\phi_0 = \beta\sigma_H(1-\pi)u'(q_{1H}) + \beta\sigma_H\pi u'(q_{1H}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta(1-\sigma_H)u'(q_{2H})$$
$$\phi_1 = \beta\sigma_H(1-\pi)u'(q_{1H})(1-\alpha_e) + \beta\sigma_H\pi u'(q_{1H}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))$$
$$\phi_0 = \beta$$
$$\phi_1 = \beta\sigma_H(1-\alpha_e)$$
$$1-\tilde{\mathfrak{p}} = \frac{\sigma_L f_L m_{L0} + \sigma_H f_H m_{H0}}{f_L m_{L0} + f_H m_{H0} + f_H m_{H1}} < 1$$

We have from the FOC of the L-buyer wrt $\mathfrak{p}_0$,

$$\begin{aligned}
\phi_0 &= \beta\sigma_L(1-\pi)u'(q_{1L}) + \beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta(1-\sigma_L)\varepsilon \\
1 &= \left[(1-\pi) + \pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]\sigma_L u'(q_{1L}) + (1-\sigma_L)\varepsilon
\end{aligned}$$
$$\frac{1-(1-\sigma_L)\varepsilon}{\sigma_L\left[(1-\pi)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]} = u'(q_{1L})$$

and we need

$$\begin{aligned}
\phi_1 &> \beta\sigma_L(1-\pi)u'(q_{1L})(1-\alpha_e) + \beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\
\sigma_H(1-\alpha_e) &> \left[\sigma_L(1-\pi)(1-\alpha_e) + \sigma_L\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]u'(q_{1L}) \\
\sigma_H(1-\alpha_e) &> \left[\frac{(1-\pi)(1-\alpha_e)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k}{(1-\pi)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k}\right](1-(1-\sigma_L)\varepsilon)
\end{aligned}$$

or

$$\sigma_H\frac{(1-\alpha_e)\left[(1-\pi)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]}{\left[(1-\pi)(1-\alpha_e)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]} > (1-(1-\sigma_L)\varepsilon)$$

Or rearranging,

$$(1-\sigma_L)\varepsilon > 1-\sigma_H\frac{(1-\alpha_e)\left[(1-\pi)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]}{\left[(1-\pi)(1-\alpha_e)+\pi(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}}))^k\right]} = 1-A\sigma_H.$$

since $A<1$ and

$$A\sigma_H < \sigma_H < \sigma_L$$

while $\varepsilon < 1$, the inequality above can never be satisfied.

Hence $\mathcal{M}_{\mathbf{H}} = \{0,1\}$,$\mathcal{M}_{\mathbf{L}} = \{1\}$ cannot be an equilibrium.

CASE (vi): Consider an equilibrium with $\mathcal{M}_{\mathbf{H}} = \{0,1\}$,$\mathcal{M}_{\mathbf{L}} = \{0,1\}$

In this case, the consumption levels are

$$
\begin{aligned}
q_{1L} &= m_{L0} + m_{L1}(1-\alpha_e) \\
q_{1L}^{\pi} &= (m_{L0} + m_{L1})\,(1-\alpha_e + \alpha_e(1-\tilde{\mathfrak{p}})) \\
q_{2L} &= m_{L0}
\end{aligned}
$$

and

$$
\begin{aligned}
q_{1H} &= m_{H0} + m_{H1}(1-\alpha_e) \\
q_{1H}^{\pi} &= (m_{H0} + m_{H1})(1-\alpha_e + \alpha_e(1-\tilde{\mathfrak{p}})) \\
q_{2H} &= m_{H0} \leq q_{1H}
\end{aligned}
$$

The equilibrium conditions are

$$
\begin{aligned}
\phi_0 &= \beta\sigma_L(1-\pi)u'(q_{1L}) + \beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta(1-\sigma_L)\varepsilon \\
\phi_1 &= \beta\sigma_L(1-\pi)u'(q_{1L})(1-\alpha_e) + \beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\
\phi_0 &= \beta\sigma_H(1-\pi)u'(q_{1H}) + \beta\sigma_H\pi u'(q_{1H}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + \beta(1-\sigma_H)u'(q_{2H}) \\
\phi_1 &= \beta\sigma_H(1-\pi)u'(q_{1H})(1-\alpha_e) + \beta\sigma_H\pi u'(q_{1H}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\
\phi_0 &= \beta \\
\phi_1 &= \beta\frac{\sigma_L\,(f_L m_{L1}) + \sigma_H\,(f_H m_{H1})}{f_L m_{L1} + f_H m_{H1}}(1-\alpha_e) = \beta\tilde{\phi}(1-\alpha_e) \\
1-\tilde{\mathfrak{p}} &= \frac{f_H m_{H0} + f_L m_{L0}}{f_L m_{L1} + f_L m_{L0} + f_H m_{H0} + f_H m_{H1}} < 1
\end{aligned}
$$

Hence,

$$
1 = \sigma_L(1-\pi)u'(q_{1L}) + \beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + (1-\sigma_L)\varepsilon
$$

and using that expression to get rid of $u'(q_{1L}^{\pi})$ in the FOC wrt $\mathfrak{p}_1$,

$$
\begin{aligned}
\tilde{\phi}(1-\alpha_e) &= \sigma_L(1-\pi)u'(q_{1L})(1-\alpha_e) + [1-\sigma_L(1-\pi)u'(q_{1L}) - (1-\sigma_L)\varepsilon] \\
\tilde{\phi}(1-\alpha_e) &= 1-(1-\sigma_L)\varepsilon - \alpha_e\sigma_L(1-\pi)u'(q_{1L}) \\
u'(q_{1L}) &= \frac{\left[1-\tilde{\phi}(1-\alpha_e) - (1-\sigma_L)\varepsilon\right]}{\alpha_e\sigma_L(1-\pi)}
\end{aligned}
$$

and now solving for $u'(q_{1L}^{\pi})$ using FOC wrt $\mathfrak{p}_0$,

$$
\begin{aligned}
1 &= \sigma_L(1-\pi)u'(q_{1L}) + \beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + (1-\sigma_L)\varepsilon \\
1 &= \sigma_L(1-\pi)\frac{\left[1-\tilde{\phi}(1-\alpha_e)-(1-\sigma_L)\varepsilon\right]}{\alpha_e\sigma_L(1-\pi)} + \beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + (1-\sigma_L)\varepsilon \\
1 &= \frac{\left[1-\tilde{\phi}(1-\alpha_e)-(1-\sigma_L)\varepsilon\right]}{\alpha_e} + \beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) + (1-\sigma_L)\varepsilon \\
\alpha_e &= \left[1-\tilde{\phi}(1-\alpha_e)-(1-\alpha_e)(1-\sigma_L)\varepsilon\right] + \alpha_e\beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) \\
\left[\tilde{\phi}+(1-\sigma_L)\varepsilon-1\right](1-\alpha_e) &= \alpha_e\beta\sigma_L\pi u'(q_{1L}^{\pi})(1-\alpha_e+\alpha_e(1-\tilde{\mathfrak{p}})) > 0
\end{aligned}
$$

This requires

$$1 < \tilde{\phi} + (1-\sigma_L)\varepsilon$$

however, since $\tilde{\phi} < \sigma_L$ this contradicts $\varepsilon < 1$.

Therefore $\mathcal{M}_{\mathbf{H}} = \{0,1\}, \mathcal{M}_{\mathbf{L}} = \{0,1\}$ cannot be an equilibrium.